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# ESSAYS ON THE THEORY OF STRUCTURAL ECONOMIC DYNAMICS — GROWTH, TECHNICAL PROGRESS, AND EFFECTIVE DEMAND

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*Nicanor parlava pacatamente con la sua voce leggermente gutturale che avevo finito per capire, e parlava di ciò che si possiede. Per centinaia di anni qui si era lavorato, ma chi possedeva, ora, l'odore di legno segato, o l'odore del lavoro? Ora si strappavano le macchine dalla segheria, e chi possedeva il lavoro che era stato fatto, chi possedeva il lavoro dello zio Aron, o di K. V., o il suo? Infine aveva capito, disse in tono pacato e paziente, come dovrebbe essere. Che il proprio lavoro non dovrebbe scomparire una volta fatto. Che il proprio lavoro finiva per diventare parte di ciò che era stato fatto. Perciò erano due secoli di sudore che ora facevano parte di quella segheria, si era lavorato insieme per costruirla, per creare quell'odore di legno. Non si poteva vendere. Non si potevano fare operazioni finanziarie, quelli che avevano lavorato qui erano diventati tutt'uno con la segheria. Il loro lavoro esisteva ancora. E così dovrebbe essere. Perché lo si sappia la prossima volta. Perché non siano derubati un'altra volta.*

Per Olov Enquist
La partenza dei musicanti
Giangiacomo Feltrinelli Editore, Milano
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# Contents

# Introduction

The initial idea from which I originally intended to develop the present dissertation was the consideration of the importance of providing empirical applications to Pasinetti's approach of structural economic dynamics, in order to show that it is not a mere intellectual exercise, providing an elegant but not very useful theoretical framework, as it has been sometimes argued by some commentators, but that it can be used to interpret and understand real, concrete economic phenomena in a way which is really, deeply alternative to the dominant one.

With this purpose in mind, I asked myself which kind of application could have been a good starting point, and which kind of empirical investigation could have been the best way of implementing it. I initially decided to try to estimate the differences between actual and 'natural' rates of profit, in the conviction that it would have been possible to evaluate the performances of a concrete economy by comparing them with the 'norm' provided by the 'natural' economic system.

Disappointingly enough, it seemed to me I could not manage to achieve this goal. How to estimate actual rates of profit? How to use national accounts data consistently with Pasinetti's original framework? Which interpretative schema to adopt and which connections to the theoretical *corpus*? Going to the roots of the problem: which was the concrete rationale of performing such an exercise, and which the correct way of performing it?

I soon understood that my standpoint was completely wrong. No doubt that skepticism about this kind of approach could only be defeated by providing a concrete example of its usage to say something about reality. No doubt that discussions with economists belonging to different 'schools of thought' could be more effectively held on the basis of some data and results at hand, the only idiom that can be easily understood by everybody. But the ground on which I was trying to walk was not firm enough.

My conclusions, in fact, were that I did not know how to fit data into Pasinetti's framework, and that even if I could have managed to do so, I did not have a completely clear view of how to present the results, and of how to persuade an eventual interlocutor about their relevance. Far away from being an insurmountable obstacle, in fact on the contrary, the answers to these questions gave to me the right clue on the direction to follow.

More precisely, the answer to these questions gave to me quite an accurate idea of what I was lacking in order to pursue my original task. The present dissertation is the result of the attempt at filling the gap, in fact representing a preparatory *theoretical* work to pave the way for the kind of investigations which at the very beginning I fondly thought I could immediately face.

First of all, I felt the necessity of re-assimilating the whole framework put forward by Pasinetti's (1981) book in a deeper conceptual way, in order to fully understand all its, many, implications and hints. Many of the issues I have then considered were not in my initial 'agenda', but came about *in itinere*, since the answer to one question often opened up a series of new ones, and so on, in a not always linear process.

To begin with, I had to dissipate the doubts concerning the meaning of the term 'pre-institutional'. How is it possible to study an industrial economic system without reference to its institutional make up? How to say something about the theory of value and *income distribution* without mentioning any specific set of *social relations of production*? How to reconcile the apparent contradiction between my own task — actually, in my opinion, the very task of political economy — i.e. studying the functioning of actual economic systems, and the principal aim of Pasinetti's (1981) book, i.e that of performing an analysis which completely abstracts from the mechanisms through which such economic systems come into being?

Answering these questions meant understanding that the above-mentioned apparent contradiction is not a contradiction at all. In order to be able to study actual economic systems, Pasinetti proposes to separate the *foundational* from the *institutional* aspect, as a way of understanding the latter in the light of the former, and thus to perform concrete investigations with a unifying interpretative schema at hand.

Pasinetti's (1981) definition of the concept of equilibrium — or better, of 'equilibrium *situation*' — is itself closely connected to the understanding of where the limit between foundational and institutional analyses lies. The object of the former, that is to say of the one developed by Pasinetti's (1981), is the *fundamental* functioning of a *capitalistic*, i.e. industrial, economic system, based on the process of production of commodities by means of commodities and on their accumulation: the task is that of singling out the *physical requirements* for its (extended) reproduction. The way in which such requirements can be met in practice pertains to the institutional stage of the analysis, that can be built by adding to the fundamental relations all the elements necessary to the description of a specific, contingent situation.

Secondly, Pasinetti's (1981) elaborations are based on a series of simplifying assumptions concerning the description of the technique in use. Such simplifications represented an obstacle to the immediate fitting of actual data into the framework. In fact, almost all inter-industry relations are ruled out, by assuming that each consumption commodity is produced by means of labour and one intermediate commodity specific to it, that enters its own, and only its own, productive capacity. Reality clearly is much more complicated.

How to overcome this difficulty? In this respect, the way had been already paved by Pasinetti himself, in two different articles: the 1973 one on 'The Notion of Vertical Integration in Economic Analysis', and the 1988 one on 'Growing subsystems, vertically hyper-integrated sectors and the labour theory of value'. Both of them make use of the complete set of inter-industry relations. Both of them consist of a re-partitioning of the activities taking place in the economic system, making direct reference to each single commodity composing the *net output*. Both of them propose the device of using a particular unit of measurement for each sector's productive capacity.

At first sight, the difference seems to consist only in the fact that the latter includes not only direct and indirect, but also hyper-indirect requirements in the description of the technique. A more careful examination, however, reveals that a deeper, *conceptual* difference does exist, going way beyond the simple inclusion of the productive effort to be put forward to increase productive capacity in the set of activities performed by each sector.

This inclusion crucially depends on a re-definition of the very concept of net output, i.e. on a different treatment of new investments, which in Pasinetti's (1988) formulation are no more part of the net output itself — which therefore comes to consist of consumption commodities only — and thus taken as *exogenous* with respect to technology, but rather produced together with, an in the same way as, all the intermediate commodities necessary to replace those used up during the production process. In Pasinetti (1973), vertically integrated sector $i$ produces the amount of commodity $i$ required by the whole economic system as additional productive capacity; in Pasinetti (1988), each vertically hyper-integrated sector $i$ produces the whole set of commodities necessary to constitute its own additional productive capacity. *This* is the difference between vertically integrated and hyper-integrated sectors.

This consciousness opened up a new question: do the sectors considered by Pasinetti (1981) belong to the former or to the latter category? Pasinetti himself is not that clear in this respect, since he uses indifferently both denominations in different parts of the book — in this way reinforcing the idea of a substantial equivalence between the two. Intense discussions and careful examination of the treatment devoted by Pasinetti (1981) to net output and new investments led me to conclude that they are vertically hyper-integrated sectors. The juxtaposition is the result of the fact that the book is an extended version of Pasinetti's doctoral dissertation (Pasinetti 1962a), where some chapters were reproduced exactly as they originally were — those talking about vertically integrated sectors — and some other chapters were written *ex novo* — those mentioning vertically hyper-integrated ones. For sure the concept was still in an embryonic stage, awaiting for its rigorous formalisation to come about with Pasinetti (1988). But the main idea was already there.

The last step to complete this conceptual *excursus* through Pasinetti's (1981) book was an examination of the characteristics and implications of the 'natural' economic system.

In the first 126 pages of his 1981 book, Pasinetti states the quantity and price systems, computes their solutions, derives the conditions for achieving flow and stock equilibrium at a single point in time, and singles out the pace at which capital accumulation has to take place in order for such an equilibrium situation to be preserved. And yet, still this is not the 'natural' economic system. We simply have a set of equilibrium solutions, one for each possible configuration of the distributive variables, i.e. one for each possible, exogenous, set of (sectoral) rate(s) of profit with which to close the price system.

Only one of these possible closures leads to the 'natural' economic system, the one — stemming from the adoption of Pasinetti's (1981) particular theory of income distribution — consisting in the 'natural' rates of profit.

It is at this point that one of my initial questions came back to the fore: how a theory of income distribution can be stated in a pre-institutional analysis? How can we define income recipients, if the categories we are used to have not yet been defined? The answer is straightforward if one has in mind the task of the foundational stage of the analysis: singling out the *physical* requirements for (extended) reproduction to take place. A necessary, even though not sufficient, condition for them to be met is the availability of the precise amount of resources that have to be devoted to capital accumulation, the residual being left for consumption. This is the key: the 'natural' profits exactly provide the economic system with the resources that must be re-injected into the production process as new investments and thus new productive capacity. The remainder of national income, wholly absorbed by wages, can be devoted to consumption. By adopting this closure of the price system, what we get are the 'natural' prices, i.e. the value counterpart of — or the exchange ratios necessary for realising — the physical-quantity equilibrium configuration. Incidentally, and interestingly enough, these exchange ratios bring with them a *pure labour* theory of value.

After completing the theoretical *excursus*, my primary aim was that of identifying the steps I needed to do in order to complete my 'preparatory work'.

First, fitting actual data into the framework required to extend the generalisation as regards the description of the technique, started by Pasinetti (1988), to the whole of Pasinetti's (1981) theoretical construction. In addition, the degree of realism could have been further increased by introducing discrete time and non-steady rates of change of the exogenous variables.

Second, the generalisation would have been better performed by making the *formulae* more compact and easy to manage. To this end, I have restated the whole analytical framework by means of (partitioned) matrix algebra, making an extended use of Perron-Frobenius theorems for non-negative matrices.

Third, and closely connected to the previous point, I needed a mathematical formulation able to make it easier to work out empirical applications, and in particular to facilitate calculations when implementing the theoretical framework with statistical software. In order to do so I have restated the quantity and price systems as eigenproblems, the solutions being the eigenvectors associated to particular eigenvalues.

The accomplishment of these three tasks constituted the majority of my dissertation. Also in this case, the process has been an almost non-linear one, during which I have made many mistakes, I have many times changed my mind, I have often thought I was going nowhere; but nonetheless, during which I have reached many new conclusions, I have explored many new points of view, and I hope to have also achieved some theoretical advance.

The dissertation is organised as a collection of four papers (the first one written in co-authorship with Ariel L. Wirkierman). Though independent, the four papers are closely connected to each other, developing through different lines a unifying argument.

The first paper — 'Pasinetti's *Structural Change and Economic Growth*: a conceptual *excursus*' (Garbellini & Wirkierman 2010b) — is a re-exposition of the framework developed in Pasinetti's (1981) book, read in the light of both: the clarification of some methodological and conceptual issues; and the contextualisation of the book within the whole intellectual path, going from 1962 to 1988, which led Pasinetti to the completion of the explicit and rigorous definition of the concept of vertically hyper-integrated sectors. The first task is accomplished through the clarification of the nature and meaning of the pre-institutional approach adopted; of the nature of its equilibrium 'paths'; and of the significance, and normative character, of the 'natural' economic system. The second aim is achieved by a historical account of Pasinetti's writings, to see how the 1981 book is an intermediate step towards the 1988 CJE article. The conceptual idea was already present very clearly in the former, though a rigorous formalisation came about only in the latter.

However, the main theoretical and empirical implications of vertical hyper-integration are still to be drawn. In order to do so, the whole theoretical construction developed in Pasinetti (1981) has to be generalised by taking advantage of the step forward represented by Pasinetti (1988). This is precisely the aim of the second and third papers of the dissertation: 'Structural Change and Economic Growth: Production in the Short Run — A generalisation in terms of vertically hyper-integrated sectors' (Garbellini 2010b); and 'Structural Change and Economic Growth: Production in the Long Run — A generalisation in terms of vertically hyper-integrated sectors' (Garbellini 2010a).

Garbellini (2010b) has three parallel aims. The first one is that of rigorously and analytically showing the contention put forward in Garbellini & Wirkierman (2010b), i.e. that Pasinetti's (1981) framework involves the same treatment of new investments and net output as that of Pasinetti (1988), therefore already dealing with vertically hyper-integrated sectors, though in a still embryonic stage. The aim is achieved through the reformulation of Pasinetti (1973) and Pasinetti (1988) in terms analytically analogous to those of Pasinetti (1981) — though using matrix algebra, and in particular *partitioned matrices* — to show the differences of the first and the analogies of the second with respect to the third.

The second one is that of restating, in all cases, the quantity and price systems as eigenproblems, to be solved by looking for a specific eigenvalue and the associated eigenvector — the macroeconomic condition being the mathematical condition for the eigenvalue we are looking for to actually be an eigenvalue of the corresponding coefficient matrices; and the solution vectors being the associated eigenvectors. The conditions for getting economically meaningful solutions out of these eigensystems are then derived and discussed.

The last aim is that of generalising the first part of Pasinetti's (1981) book, i.e. that devoted to production in the short run, by removing Pasinetti's (1981) simplifying assumptions as to the description of the technique in use — i.e. by using the complete inter-industry matrix as in Pasinetti (1988) — in order to deepen the analysis and make all the theoretical categories directly comparable with the empirical ones, as coming from national accounts. In this way, the ground should be prepared for implementing empirical applications, deepening the analysis by considering the whole set of inter-industry relations, and extending the generalisation to the most important aspect of vertical hyper-integration, that is to say the one concerning dynamics, and thus production in the long run.

This last task is performed by the third chapter of the dissertation, Garbellini (2010a). In this paper I restate the laws of motion of all the exogenous variables — intended by Pasinetti (1981) as exponential functions of their initial values, changing through continuous time at steady, though different from sector to sector, rates — in discrete time, thus introducing non-steady rates of change. It is my contention that this is an improvement with respect to the dynamics assumed in the original formulation, since it increases the degree of realism of the whole framework, allowing to single out determinants of the structural change of the economic system that cannot be identified by using exponential growth with steady rates of change.

By using these discrete dynamics — besides reconsidering the determinants of the structural dynamics of quantities and prices, as well as of the sectoral and aggregate capital/output and capital/labour ratios — I then restate the conditions for keeping stock equilibrium through time, i.e. Pasinetti's capital accumulation conditions, stressing their relations with the rate(s) of profit. This, together with the sketching of the particular theory of income distribution used by Pasinetti to get a closure of the price system (already detailed in Garbellini & Wirkierman 2010b) allows to define the *natural rates of profit*, and therefore the 'natural' economic system. The characteristics of the latter, in particular as to the peculiarities of the value formation side of the economic system itself and thus of the resulting theory of value, are then analysed and discussed in detail. The last part of the paper is devoted to the restatement of Pasinetti's (1981) 'standard rate of growth of productivity' and thus 'dynamic standard commodity', and to resume the argument leading to the definition of the 'natural rate of interest', that has remained somehow unnoticed and obscure after its original statement in Pasinetti (1981).

Finally, the last chapter of the dissertation (Garbellini 2010c) aims at putting together the results of the first three chapters in order to provide a reply to the criticisms more often put forward against the approach of structural economic dynamics.

As a matter of conclusion, I would like to stress the fact that I did not make, in the present Introduction, any reference to the superiority of the approach of structural economic dynamics — and, more in general, of the 'modern classical' approach to economic analysis — with respect to mainstream economics. That famous controversy finished with an undoubted victory of the former against the latter. I personally consider it as one of the most fascinating pages of the history of economic thought. Though, unfortunately, nothing changed afterwards. For this reason, I am absolutely convinced that today, after half a century, it is necessary to keep up working on the *constructive* side, recovering what I consider the real strength of the 'production paradigm': a theorising process based on observable, measurable, concrete categories, closely connected to national accounting, constantly aiming at explaining reality. Deeply concerned with social matters. Not economic science, but Political Economy.

# Pasinetti's *Structural Change and Economic Growth*: a conceptual *excursus*[*]

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**Abstract** A clear and organic exposition of Pasinetti's theoretical framework of *Structural Change and Economic Growth* is often complicated by misunderstandings and ambiguities concerning the basic categories and terminology.

The pre-institutional character of the approach, the nature of its equilibrium paths and the significance of the 'natural' economic system — together with its normative character — are some of the most controversial issues.

In particular, there seems to be a need for a clearcut distinction between the general dynamic analysis of the price and quantity systems and the specific dynamics they follow when the sectoral proportions and levels of production exactly satisfy *dynamic equilibrium* conditions, and a particular closure of the price system is adopted, providing for specific *functional* income distribution and theory of value.

The aim of the present paper is therefore that of attempting at a conceptual *excursus* of the model, in order to establish a solid ground on the basis of which discussions with other *Classical* approaches can be fruitfully held.

**Keywords** Vertically (hyper-)integrated sectors, functional income distribution, 'natural' economic rates of profit, 'natural' economic system, pure labour theory of value.

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# 1 Introduction

Pasinetti's *Structural Change and Economic Growth* has been, since its publication in 1981, the object of many reviews and comments, and it is one of the most cited works as regards the topic of structural change (see for example Silva & Teixeira 2008).

However, many aspects of the book, both conceptual and analytical, have not been grasped, or have been grasped only partially, thus preventing from a complete understanding of the implications, and potentialities, of Pasinetti's approach.

The first stumbling block has usually been the pre-institutional character of the model, sometimes misinterpreted as a pre-industrial one.

A second problem is the often missed distinction between the general dynamic analysis of the price and quantity systems, the dynamic equilibrium paths — one for each possible *exogenous* combination of distributive variables — and the 'natural' economic system, resulting from a particular closure of the price system.

A third issue of importance is the vertically hyper-integrated character of the framework, on which we particularly insist in the paper, in order for the model — and some of its most far reaching insights — to be fully understood.

The present paper consists of a conceptual *excursus* of the model, and is organised as follows. Section 2 deals with the pre-institutional character of Pasinetti's (1981) model. Section 3 presents a synthetic exposition of the model, *before* the introduction of the 'natural' economic system. Section 4, then, gives the rationale for the particular closure of the price system adopted by Pasinetti (1981), highlighting some of the main insights. Section 5 is a methodological note on the notion of equilibrium and its role throughout the analysis. Finally, section 6 goes through the stages of development of the concept and analytical device of vertical hyper-integration.

# 2 The pre-institutional analysis of an industrial system

Before going into the details of Pasinetti's (1981) analytical formulation, it is worth putting forward a brief methodological introduction, in order for "this theoretical framework [...] [to be] appropriately understood and correctly used. It is a basic framework, a skeleton, so to speak, which is meant to remain at a *pre-institutional level of investigation*" (Pasinetti 1985, p. 274, italics added).

The above excerpt comes from the reply Pasinetti gave to a review, by Nina Shapiro (1984), of *Structural Change and Economic Growth.* The point he raised is crucial: the analytical framework he developed can be understood and correctly used only if its pre-institutional character is constantly and clearly kept in mind.

Therefore, even if the deepest implications of Pasinetti's (1981) methodology will be drawn later on, in section 5 below, a general hint must be given here, before going into the analytical description of the model, in order for the latter to be properly understood.

The first thing that should be made clear is the meaning Pasinetti attaches to the word 'capitalistic', as opposed to 'capitalist'. While the latter refers to the set of *social relations* of production typical of a capitalist economic system, in contrast to those typical, for example, of a centrally planned one, the former term describes the very physical-technological nature of the production process in any *industrial* system, to be intended as the *production of commodities by means of commodities*, i.e. the employment of capital goods as intermediate commodities, to be used together with labour, and accumulated, for the production process to take place.

Pasinetti's focus has *always* been, in all his works, on "industrial societies, with their tendency towards change and towards an evolving structure, as against the more static conditions of pre-industrial societies" (Pasinetti 2005, p. 247). Nonetheless, the pre-institutional analysis he puts forward has sometimes been (mis)interpreted as a pre-capitalist, or pre-capitalistic, one, in spite of the fact that he has never used such expressions, and that he has always made explicit, and repeated, reference to 'pure production systems'.

Anyway, Pasinetti's aim is that of analysing the working of a *capitalistic*, and not of a capitalist, economic system. As will be further discussed in section 5 below, the framework he develops is by no means an attempt at describing the functioning of an actual capitalist system. Nor it is an attempt at describing the functioning of a centrally planned economy, as someone could have been induced to conclude.

What does it mean, therefore, that Pasinetti's (1981) analysis has been carried out at the *pre-institutional* level?

The issue is not a trivial one. Quite apparently, many commentators did not succeed in grasping the meaning of such a statement, thereby failing to grasp the very nature of Pasinetti's framework. It is our contention that many — actual or pretended — ambiguities in Pasinetti's (1981) expositions are due to this misunderstanding.

As Pasinetti states in the Introduction of *Structural Change and Economic Growth*, his approach to economic theory starts from a very precise standpoint:

> It is my purpose [...] to develop first of all a theory which remains neutral with respect to the institutional organisation of society. My preoccupation will be that of singling out, to resume Ricardo's terminology, the 'primary

> and natural' features of a pure production system.
>
> (Pasinetti 1981, p. 25)

A 'separation' — as Pasinetti called it later, in his most recent book[1] — is therefore needed between two stages of analysis, each concerning a very specific kind of economic investigation. The rationale of this separation emerges very clearly from the Preface to the 1981 book:

> There is [...] a sharp discrimination between those economic problems that have to be solved on the ground of logic alone — for which economic theory is entirely autonomous — and those economic problems that arise in connection with particular institutions, or with particular groups' or individuals' behaviour — for which economic theory is no longer autonomous and needs to be integrated with further hypotheses, which may well come from other social sciences. It is with the first type of problems that the present work is basically concerned.
>
> (Pasinetti 1981, p. xiii)

Of course, Pasinetti's (1981) claim for the *logical* priority of the first stage — the *pre-institutional* one — with respect to the second stage — the *institutional* one — by no means implies that he is disregarding the role of institutions. On the contrary: their role is regarded as one of primary importance, as institutions are the *means* through which it is possible to shape the real world:

> All these considerations only come to confirm how important is to keep the logical problems concerning the 'natural' economic system quite separate from those concerning the institutions, and to consider the institutions they really are — means, and not ends in themselves. Once their instrumental role is properly understood and recognised, it becomes much easier also to operate on them in as detached a way as is possible; to treat them as instruments susceptible to be continually improved and changed, in relation to their suitability (or unsuitability) to ensure tendencies, or near-tendencies, towards agreed ends.
>
> (Pasinetti 1981, p. 155)

Institutions are means, not ends, but in order for them to be used to drive society 'towards agreed ends' it is first of all necessary to know the fundamental mechanisms they are called upon to counteract, or to favour, or simply to take advantage of. Without this knowledge, institutions cannot pursue any instrumental role.

Pasinetti's *vision* is that the 'primary and natural' features of an economic system have to be studied independently of a particular institutional set-up. Nonetheless, the task of describing an economic system without reference to a particular

[1] Pasinetti (2007), where he stresses in a much sharper way such a discrimination.

institutional set-up is not a trivial one. It is very difficult to realise how an economic system can be thought of without strong reference to the institutions which shape it, since no actual economic system could have been brought into existence without them.

This task can be accomplished by looking for those *physical* requirements necessary for an industrial system to carry out its production process, and *grow*. The way in which Pasinetti puts this idea into practice shall become clear by reading sections 3 and 4, below.

# 3 General dynamic analysis and equilibrium dynamics

Pasinetti's (1981) *Structural Change and Economic Growth* provides us with a model of economic growth starting from a complete description of an economic system in a single-period *equilibrium*, defined as "a situation in which there is full employment of the labour force and full utilisation of the existing productive capacity" (Pasinetti 1981, pp. 48-49). This situation can be thought of as the *initial condition* of a general multi-sector dynamic model, which "has been developed for the purpose of detecting the 'permanent' causes moving an economic system, irrespective of any accidental or transitory deviation which may temporarily occur" (Pasinetti 1981, p. 127).

We will now introduce a synthetic exposition of the model.[2] For basic notation see Table 1.

## 3.1 Formulation of quantity and price systems

This single-period description consists of a physical quantity system and a commodity price system, each composed by $2m+1$ equations, where $m$ is the number of final consumption commodities produced in the system. The production of each consumption commodity $i$ requires a specific capital good $k_i$. As to the physical quantity system, this means that the equation concerning consumption commodity $i$ and the equation concerning the corresponding capital good $k_i$ together describe the total quantities produced by the vertically hyper-integrated sector[3] $i$ ($x_i(t)$ and $x_{k_i}(t)$, $i = 1, 2, \ldots, m$). The last equation establishes the condition for the full

[2] We will expose here the specification of the model that considers capital goods produced by means of labour alone, for it is the main case Pasinetti (1981) deals with.

[3] The concept of vertically hyper-integration is already present in Pasinetti (1981), even though not always explicitly. For a rigorous statement and development of this concept, and of its analytical properties, see Pasinetti (1988).

Table 1: Basic notation in Pasinetti's *Structural Change and Economic Growth*

| | |
|---|---|
| $x_i(t)$ | number of units of final consumption commodity $i$ produced during time period $t$ in the vertically hyper-integrated sector $i$; |
| $x_{k_i}(t)$ | gross investment in the vertically hyper-integrated sector $i$, i.e. number of units of productive capacity for final consumption commodity $i$ produced during time period $t$; |
| $x_n(t)$ | total units of labour available at the beginning of time period $t$; |
| $p_i(t)$ | price of a unit of final consumption commodity $i$ during time period $t$; |
| $p_{k_i}(t)$ | price of a unit of productive capacity for final consumption commodity $i$ during time period $t$; |
| $w(t)$ | wage rate during time period $t$; |
| $\pi_i(t)$ | profit rate of the industry producing final consumption good $i$ during time period $t$; |
| $a_{in}(t)$ | average per capita demand for final consumption commodity $i$ during time period $t$; |
| $a_{k_in}(t)$ | average per capita demand for units of productive capacity for final consumption commodity $i$ during time period $t$; |
| $a_{ni}(t)$ | direct labour requirements for the production of one unit of final consumption commodity $i$ during time period $t$; |
| $a_{nk_i}(t)$ | direct labour requirements for the production of one unit of productive capacity for final consumption commodity $i$ during time period $t$; |
| $T_i$ | reciprocal of the coefficient of wear and tear of one unit of productive capacity for final consumption commodity $i$; |
| $x'_{k_i}(t)$ | demand for units of productive capacity for final consumption commodity $i$ for replacement of worn out capacity during time period $t$; |
| $x''_{k_i}(t)$ | net investment in the vertically hyper-integrated sector $i$, i.e. new investment demand for units of productive capacity for final consumption commodity $i$ during time period $t$; |
| $k_i(t)$ | stock of units of productive capacity for the vertically hyper-integrated sector $i$ available at the beginning of time period $t$; |
| $\chi_i(t)$ | capital/output ratio at current prices in time period $t$ for vertically hyper-integrated sector $i$; |

employment of the total labour available in the system:

$$\begin{cases} x_i(t) - a_{in}(t)x_n(t) = 0 & \text{for} \quad i = 1, 2, \ldots, m \\ T_i^{-1}x_i(t) - x_{k_i}(t) - a_{k_i n}(t)x_n(t) = 0 & \text{for} \quad i = 1, 2, \ldots, m \\ \sum_i a_{ni}(t)x_i(t) + \sum_i a_{nk_i}(t)x_{k_i}(t) - x_n(t) = 0 \end{cases} \tag{3.1}$$

As to the price system, there will be a price for each final consumption commodity $i$ and a price for each capital good $k_i$ associated to it ($p_i(t)$ and $p_{k_i}(t)$, $i = 1, 2, \ldots, m$). The last equation establishes the condition for the full expenditure of (full employment) national income:

$$\begin{cases} -p_i(t) + \left(\pi_i(t)\frac{k_i(t)}{x_i(t)} + \frac{1}{T_i}\right) p_{k_i}(t) + a_{ni}w = 0 & \text{for} \quad i = 1, 2, \ldots, m \\ -p_{k_i}(t) + a_{nk_i}(t)w(t) = 0 & \text{for} \quad i = 1, 2, \ldots, m \\ \sum_i a_{in}(t)p_i(t) + \sum_i \left(a_{k_i n}(t) - \pi_i(t)\frac{k_i(t)}{x_i(t)}\right) p_{k_i}(t) - w(t) = 0 \end{cases} \tag{3.2}$$

## 3.2 Vertically hyper-integrated productive capacity

In general, the means of production required to obtain one unit of a final consumption good are a sector-specific composite commodity in which the same intermediate inputs enter in — technically given — proportions. This motivates the definition of a particular unit of measurement — one for each vertically hyper-integrated sector — for this physical composite commodity, a *unit of vertically hyper-integrated productive capacity*. Each of these units is the sum of three components: direct requirements for the production of one unit of final consumption commodity $i$; direct requirements for the replacement of worn-out direct and indirect capital goods needed for the production of one unit of final consumption commodity $i$; and direct requirements for the production of all intermediate commodities directly and indirectly needed for the expansion of productive capacity in line with the growth of final demand for consumption commodity $i$.

In Pasinetti's (1981) theoretical scheme, the units of productive capacity used as units of measurement for capital goods are actually units of *direct* productive capacity for the production of final consumption commodities. This becomes clear when looking at the most complex case, in which capital goods are produced by means of labour *and* capital goods (see Pasinetti 1981, pp. 43-45). However, it is our contention not only that units of vertically hyper-integrated productive capacity are the most appropriate units of measurement for capital goods, but also that Pasinetti himself, in 1981, had already begun to argue in terms of vertically hyper-integration, even if the complete analytical implications were still to be reached (many of them finally reached their rigorous formulation with the publication of Pasinetti (1988)).[4]

[4]This is particularly clear when matching the chapters of the book which "have been

Anyway, in the present, simpler, case, no analytical difference can be found between direct, vertically integrated and vertically hyper-integrated productive capacity, since final consumption commodities are the only ones produced by means of capital goods. Therefore we can interpret units of productive capacity as being vertically hyper-integrated, without having to reformulate the analytical model in Pasinetti (1981). For the sake of simplicity, from now on, we will simply say 'sectors' instead of 'vertically hyper-integrated sectors'; and 'productive capacity' instead of 'vertically hyper-integrated productive capacity', except where the complete expressions shall be considered more appropriate.

A unit of productive capacity will refer to the specific final commodity that requires it, and therefore to the specific sector in which it is produced. In this way, the analysis opens up for the possibility of separating the pace of accumulation of the means of production (the number of units of productive capacity) from its physical composition.

In order to simplify exposition, Pasinetti (1981) regards these composite commodities as particular capital goods, specific to each consumption good. Therefore, as hinted at above, in the present context, a vertically hyper-integrated sector is made up by two industries: one producing the final consumption good, and the other one producing the corresponding capital good. These two industries play an asymmetric role, since "the physical quantities of the means of production appear as playing a sort of ancillary role with respect to the physical quantities of final demand [for consumption goods]; the former being, so to speak, 'at the service' of the latter" (Pasinetti 1988, pp. 125-126).

## 3.3 Conditions for flow and stock-equilibrium solutions

Both physical quantity and commodity price systems are formulated as sets of $2m+1$ linear and homogeneous equations which have non-trivial solutions if the coefficient matrix is singular, i.e. if its determinant is zero. The condition for this to be true is the same for both equation systems, and can be written as:

$$\sum_i a_{in}(t)a_{ni}(t) + \sum_i T_i^{-1} a_{in}(t)a_{nk_i}(t) + \sum_i a_{k_i n}(t)a_{nk_i}(t) = 1 \tag{3.3}$$

If this condition is not satisfied, the systems are contradictory, i.e. each set of $2m+1$ equations cannot simultaneously hold. However, because of the particular mathematical structure of the problem,[5] we can still get meaningful solutions for

almost entirely re-written" (Pasinetti 1981, p. xiv) since the time of his PhD Thesis with the entries in the index concerning vertical *hyper*-integration. We shall come back to this point later on, in section 6.

[5]For details, see Pasinetti (1981, pp. 33-34).

quantities and prices, but the last equation in each system will not be satisfied, i.e. we shall not be in a situation of full employment of the labour force and full expenditure of total income. On the contrary, if this single condition is satisfied, the solutions will correspond to a situation of full employment and full expenditure of income. To this situation we shall refer as a *flow-equilibrium* situation.

Assuming that condition (3.3) holds, we get two indeterminate linear homogeneous systems, which means that we have solutions for *relative* quantities and *relative* prices corresponding to a situation of flow-equilibrium, but we have to choose a scale factor for each system. For the quantity system this scale factor is total labour available, since this is an exogenous variable. Therefore, Pasinetti sets $x_n(t) = \overline{x}_n(t)$. For the price system, the choice of a scale factor is in fact an arbitrary one. Following Pasinetti (1981, pp. 92-93), we choose, for convenience, the wage rate, and therefore we take it as given both at a specific point in time and through time, i.e. we set $w(t) = \overline{w}$.

Therefore, for a given period $t$, the solutions for physical quantities and commodity prices — in a *flow* equilibrium — are, respectively:[6]

$$\begin{cases} x_i(t) = a_{in}(t)\overline{x}_n(t) \\ x_{k_i}(t) = T_i^{-1}a_{in}(t)\overline{x}_n(t) + a_{k_in}(t)\overline{x}_n(t) \end{cases} \tag{3.4}$$

and

$$\begin{cases} p_i(t) = \left(a_{ni}(t) + a_{nk_i}(t)\dfrac{k_i(t)}{x_i(t)}(T_i^{-1} + \pi_i(t))\right)\overline{w} \\ p_{k_i}(t) = a_{nk_i}(t)\overline{w} \end{cases} \tag{3.5}$$

for $\quad i = 1, 2, \ldots, m$

Expressions (3.4) and (3.5) are made up by $m$ pairs of equations each. The first equation of each pair concerns consumption commodity $i$ ($i = 1, 2, \ldots, m$); the second concerns the corresponding unit of vertically hyper-integrated productive capacity $k_i$ ($i = 1, 2, \ldots, m$).

Expressions (3.4) represent solutions for total physical quantities produced in the vertically hyper-integrated sectors $i = 1, 2, \ldots, m$. First, $x_i$ is determined by the average per capita demand for final consumption commodity $i$ multiplied by total population. Second, $x_{k_i}$ is the sum of two components: $T_i^{-1}a_{in}(t)\overline{x}_n(t) =$

[6]In the solution for $p_i(t)$ given in Pasinetti (1981, p. 41) it is implicitly assumed that $x_i(t) = k_i(t)$:

$$p_i(t) = \left(a_{ni}(t) + a_{nk_i}(t)(T_i^{-1} + \pi_i(t))\right)\overline{w}$$

This amounts to stating that the productive capacity available at the beginning of time period $t$ is exactly used up. In order to make the formulation as general as possible, we have decided not to make this assumption at this stage of the analysis.

$x'_{k_i}(t)$, i.e. the number of units of productive capacity for consumption good $i$ necessary for the replacement of worn out productive capacity, and $a_{k_in}(t)\overline{x}_n(t) = x''_{k_i}(t)$, i.e. the number of units of productive capacity demanded as new investment. The capital-producing industry has to produce not only those units of productive capacity necessary for keeping the initial stock of (units of) productive capacity intact, but also those units required to expand it.[7]

Expressions (3.5) are the solutions for commodity *production prices*. Since we are assuming that capital goods are produced by labour alone, each price $p_{k_i}$ is determined by its direct labour requirements multiplied by the wage rate, while in the expressions for prices $p_i$ the wage rate also multiplies a gross profit mark-up proportional to the direct labour required to produce a unit of the corresponding productive capacity.

In principle, there is no difference between production prices obtained when the price system is formulated in terms of industries and when it is formulated in terms of sectors. The technique in use and the distributive variables do not change as a consequence of adopting the procedure of vertical hyper-integration, which is simply a way of re-classifying and partitioning activities in order to explicitly acknowledge for the relationship between each activity producing a final consumption commodity and those activities producing the means of production for self-replacement and expansion of the corresponding productive capacity.[8]

The difference is introduced as a consequence of the specific adoption of the units of productive capacity for final consumption commodities as the units of measurement of capital goods. In this way, each price $p_{k_i}$ does not stand for the price of one 'ordinary unit' of commodity $k_i$, but for the price of one unit of productive capacity for consumption good $i$.

As stated above (section 3, page 13), an equilibrium position entails full employment of the total labour available — which implies a *single* condition concerning *flows* — and full utilisation of the existing productive capacity in each vertically hyper integrated sector $i$ $(i = 1, 2, \ldots, m)$ — a series of sectoral conditions concerning *stocks*.

The condition concerning the flows of the economic system has already emerged as the condition for non-trivial solutions to the quantity and price systems, i.e. expression (3.3), which is a *macroeconomic* condition, since it refers to the economic system as a whole, no matter how many sectors there are. Moreover, "it emerges from a model which has been developed on a *multi-sector* basis, thereby revealing its truly macro-economic nature" (Pasinetti 1981, p. 35).

As concerns stocks, we have a *series* of sectoral conditions, saying that in each

[7]This is a crucial difference between the notion of vertically integrated sectors and that of vertically *hyper*-integrated sectors (see Pasinetti 1973, Pasinetti 1988).

[8]See Pasinetti (1973, p. 7, section 5) and Pasinetti (1988, p. 130, section 4).

sector, the number of units of productive capacity available at the beginning of the period as the capital stock endowment must be exactly equal to the number of units of final consumption good to be produced during the same time period, i.e.:

$$k_i(t) = x_i(t), \qquad i = 1, 2, \ldots, m \tag{3.6}$$

Expressions (3.4) and (3.5) together with conditions (3.3) and (3.6) exhaust the description of single-period equilibrium which is the starting point for the development of the general multi-sector dynamic model of growth.

For the analysis we are going to perform, we shall assume, from now on, that the economic system starts from a situation of both flow and stock equilibrium, i.e. we shall assume that, for time $t = 0$, both condition (3.3) and the series of conditions (3.6) hold true.

## 3.4 General dynamic analysis

The dynamic method Pasinetti (1981) adopts is that of specifying exponential laws of movement for the coefficients in (3.1) and (3.2) concerning total available labour ($x_n$), average per capita demand ($a_{in}$), and labour input requirements ($a_{ni}$ and $a_{nk_i}$), according to:[9]

$$\begin{cases} x_n(t) = x_n(0)e^{gt} \\ a_{in}(t) = a_{in}(0)e^{r_i t} \\ a_{ni}(t) = a_{ni}(0)e^{-\varrho_i t} \\ a_{nk_i}(t) = a_{nk_i}(0)e^{-\varrho_{k_i} t} \end{cases} \qquad i = 1, 2, \ldots, m \tag{3.7}$$

Solutions (3.4) and (3.5) are therefore linear structures whose components follow exponential dynamics. Taking expressions (3.4) and (3.5) evaluated at time period $t = 0$, and inserting the dynamics described in (3.7) we obtain the following

[9]For the sake of simplicity, we are here assuming steady rates of change of the relevant variables, though this is not the procedure adopted by Pasinetti (1981), at least for the rate of change of final demand for consumption commodities (See Pasinetti 1981, p. 82). This is a crude simplification, though it is not possible — according to the authors — to take full advantage of the increasing realism of working with non-steady rates of change if the model is specified in continuous time. For the scope of the present work, moreover, the simplification adopted does not compromise the conclusions to be reached.

solutions for physical quantities and commodity prices:

$$\begin{cases} x_i(t) = a_{in}(0)\overline{x}_n(0)e^{(g+r_i)t} \\ x_{k_i}(t) = T_i^{-1}a_{in}(0)\overline{x}_n(0)e^{(g+r_i)t} + a_{k_in}(t)\overline{x}_n(0)e^{gt} \end{cases} \tag{3.8}$$

and

$$\begin{cases} p_i(t) = \left(a_{ni}(0)e^{-\varrho_i t} + a_{nk_i}(0)\frac{k_i(t)}{x_i(t)}(T_i^{-1} + \pi_i(t))e^{-\varrho_{k_i}t}\right)\overline{w} \\ p_{k_i}(t) = a_{nk_i}(0)e^{-\varrho_{k_i}t}\overline{w} \end{cases} \tag{3.9}$$

for $\quad i = 1, 2, \ldots, m$

The dynamic movements in (3.8) and (3.9) do not imply full employment and full utilisation of productive capacity *after* time period $t = 0$. In particular, full utilisation of productive capacity depends on a series of *stock* conditions. In the present model, the stocks of the economic system change according to the flow of demand for new investment, linking one period to the following one. The accounting identity that describes this process of capital accumulation in each sector is:

$$\dot{k}_i(t) \equiv x''_{k_i}(t), \qquad i = 1, 2, \ldots, m \tag{3.10}$$

Given that $x''_{k_i}(t) = a_{k_in}(t)x_n(t)$, we obtain $\dot{k}_i(t) = a_{k_in}(t)x_n(t)$. Therefore, the series of coefficients $a_{k_in}(t)$ "is the only one that affects the stocks of the economic system, i.e. productive capacity in each sector; hence it cannot be taken as given from outside" (Pasinetti 1981, p. 85).

This opens up for the possibility to perform a general dynamic analysis by specifying a law of movement for the level of per capita new investment demand ($a_{k_in}$), allowing for the discrepancy between productive capacity available at the beginning of period $t$ ($k_i$) and the units of productive capacity actually used up during period $t$ ($x_i$). The specification of the dynamics of investment is a degree of freedom that, once closed, allows for performing an *institutional* analysis of different theories of capital accumulation.

Another degree of freedom can be opened by changing the last equation of both the physical quantity and the commodity price systems, in order to explicitly allow for the possibility of flow-disequilibrium, e.g. by writing:

$$\sum_i a_{ni}(t)x_i(t) + \sum_i a_{nk_i}(t)x_{k_i}(t) - \alpha x_n(t) = 0 \tag{3.11}$$

and

$$\sum_i a_{in}(t)p_i(t) + \sum_i \left(a_{k_in}(t) - \pi_i(t)\frac{k_i(t)}{x_n(t)}\right)p_{k_i}(t) - \alpha w(t) = 0 \tag{3.12}$$

where $\alpha \gtreqless 1$. This will accordingly modify the condition for non-trivial solutions to exist, which will become:

$$\sum_i a_{in}(t)a_{ni}(t) + \sum_i T_i^{-1} a_{in}(t)a_{nk_i}(t) + \sum_i a_{k_in}(t)a_{nk_i}(t) = \alpha \gtreqless 1 \qquad (3.13)$$

meaning that macroeconomic condition (3.3) is not satisfied if $\alpha \neq 1$.[10]

## 3.5 Dynamic equilibrium conditions and vertical hyper-integration

Since Pasinetti's (1981) theoretical scheme aims at describing "the 'primary and natural' features of a pure production system [...] [which] will simply emerge as necessary requirements for equilibrium growth" (Pasinetti 1981, p. 25), he is concerned, on the one hand, with the condition for keeping full-employment through time (*flow-equilibrium*), and on the other hand, with the condition for maintaining full utilisation of productive capacity through time (*stock-equilibrium*).

As regards the flow-equilibrium, by inserting (3.7) into (3.3) we get:

$$\sum_i a_{in}(0)a_{ni}(0)e^{(r_i-\varrho_i)t} + \sum_i \frac{1}{T_i} a_{in}(0)a_{nk_i}(0)e^{(r_i-\varrho_{k_i})t} + \sum_i a_{k_in}(t)a_{nk_i}(0)e^{-\varrho_{k_i}t} = 1 \qquad (3.14)$$

It can be noticed that the demand coefficients for new investment are still taken as exogenously given, while their specification will be the subject of the following few paragraphs.

As regards the stock-equilibrium, the laws of motion of average per capita sectoral demands for new investment "must be such as to be compatible with the process of economic growth and will therefore themselves be determined as part of the equilibrium conditions" (Pasinetti 1981, p. 85).

Therefore, as $a_{k_in}(t)x_n(t) = \dot{k}_i(t)$ and, in stock equilibrium, $k_i(t) = x_i(t)$, demand for new investment must exactly satisfy the growth requirements of productive capacity in each sector, as determined by the growth of demand for each final consumption commodity ($\dot{x}_i(t) = (g + r_i)a_{in}(t)x_n(t)$) in all periods beyond $t = 0$, i.e. the following set of sectoral *capital accumulation conditions* must be satisfied:

$$a_{k_in}(t) = (g + r_i)a_{in}(t), \quad \forall i = 1, 2, \ldots, m; \quad t \geq 0 \qquad (3.15)$$

which are the dynamic counterpart of stock-equilibrium conditions (3.6), stating the sectoral equilibrium *rates of new investment* — $(g+r_i)$ — defined as the number of units of productive capacity, per unit of final demand for each consumption

[10] For a hint at different cases that can occur as a consequence of flow and stock disequilibria, see Pasinetti (1981, pp. 47-48).

commodity $i$, necessary as new investment for the expansion of the corresponding productive capacity.

The series of conditions (3.15) can also be expressed as ratios of sectoral new investment to production, at current prices:

$$\frac{p_{k_i}(t)x''_{k_i}(t)}{p_i(t)x_i(t)} = (g + r_i)\frac{p_{k_i}(t)k_i(t)}{p_i(t)x_i(t)} \equiv (g + r_i)\chi_i(t) \tag{3.16}$$

When written in this way, sectoral conditions (3.15) tell us that, in stock-equilibrium, "the ratio of new investments to the level of production must be equal, in each sector, to the technologically determined capital/output ratio [$\chi_i(t)$] multiplied by [the sum of] the rate of population growth [and the rate of growth of per capita demand]" (Pasinetti 1981, pp. 54-55).

In order to fully acknowledge for the importance of conditions (3.16), we have first to notice the *vertically hyper-integrated character* of capital/output ratio $\chi_i(t)$. In a traditional inter-industry scheme, the net output of the economy is the set of commodities produced for final consumption and new investment. However, in a vertically hyper-integrated framework, the net output of the system is made up only of the set of commodities for final consumption, as new investment demand is part of the means of production required to expand productive capacity. Therefore, when thinking of the capital intensity of a sector $i$, its denominator (the net output) will be the value of final consumption commodity $i$ produced in the system, while its numerator (the value of capital) will be the value of the units of productive capacity specific to each final consumption commodity $i$ required to self-replace and expand the productive capacity during period $t$.

In the light of this, the specification of an equilibrium schedule of capital accumulation in vertically hyper-integrated terms reflects, on the one side, the interdependent nature of the production process, as — in the most general case — a single industry producing a basic commodity utilised as a capital good would participate in different sectors with a different capital intensity in each of them; and on the other side, it highlights the potential of working with vertically hyper-integrated sectors, as "the notion of a physical unit of productive capacity, by being defined with reference to the commodity that is produced, continues to make sense, as a physical unit, whatever complications technical change may cause to its composition in terms of ordinary commodities" (Pasinetti 1973, p. 24).[11]

This is the most remarkable property of the chosen unit of measurement: whatever the time period, whatever the stage of technical progress, whatever the tech-

[11] In Pasinetti (1981), as each capital goods-producing industry is specific to each consumption goods-producing one, it is the *second* aspect that is emphasised, though the framework allows for further generalisation to reflect also the first one. See Pasinetti (1988).

nique actually in use, capital goods can always be measured in units of productive capacity, and the accumulation of capital can always be studied by evaluating the *number* of units of productive capacity that have to be produced during time period $t$ to maintain stock-equilibrium at the beginning of time period $t+1$. In this way, we can link the stocks of different time periods through the simple capital accumulation (equilibrium) conditions (3.15) — or, equivalently, (3.16). Complementarily, the problem of the change in the *physical composition* of these units can be studied separately by exploiting the one-to-one correspondence between vertically hyper-integrated and inter-industry relations as "the production coefficients of a vertically [hyper-]integrated model turn out to be a linear combination of the production coefficients of the corresponding input-output model"(Pasinetti 1981, p. 111).

By substituting capital accumulation conditions (3.15) into the macroeconomic condition (3.14) and writing it as follows:

$$\sum_i a_{ni}(0)a_{in}(0)e^{(r_i-\varrho_i)t} + \sum_i \left(g + r_i + T_i^{-1}\right) a_{nk_i}(0)a_{in}(0)e^{(r_i-\varrho_{k_i})t} = 1 \qquad (3.17)$$

we can notice that the two addenda distribute total labour of the system between the labour requirements of final consumption commodities and the labour requirements of equilibrium *gross* investments.

By looking at expression (3.17), it is immediately clear that, for any specific *composition* of final demand *for consumption*, the equilibrium amount of gross investment is univocally determined by the technique in use and by the dynamics of population and of final consumption demand itself. Hence, the left-hand side of (3.17) stands for the size of per-capita total effective demand in time period $t$. Therefore, (3.17) "may be called the *effective demand condition* for keeping full employment" (Pasinetti 1981, p. 54), since it establishes whether a given composition of final demand for consumption is compatible with flow-equilibrium, i.e with full-employment of the labour force.

It therefore follows that "the difficulty of increasing total effective demand is one of finding out, and achieving, at a sufficient speed, its appropriate structural composition, and not one of reaching any absolute level" (Pasinetti 1981, p. 242), highlighting the multi-sectoral foundation of an effective demand theory of output.

## 3.6 Vertically hyper-integrated labour

The units of productive capacity are one of the two constituent components of the technique of a vertically hyper-integrated sector, the other one being the *vertically hyper-integrated labour coefficients*. In order to define them, we shall start from the full-employment macroeconomic condition for flow-equilibrium. By inserting

(3.15) into (3.14) and rearranging, we get:

$$\sum_i a_{in}(0)e^{r_i t}\left(a_{ni}(0)e^{-\varrho_i t}+\frac{1}{T_i}a_{nk_i}(0)e^{-\varrho_{k_i}t}+(g+r_i)a_{nk_i}(0)e^{-\varrho_{k_i}t}\right)=1 \quad (3.18)$$

which, by defining

$$\ell_i(t)\equiv a_{ni}(t)+\frac{1}{T_i}a_{nk_i}(t)+(g+r_i)a_{nk_i}(t) \quad (3.19)$$

can be written as:

$$\sum_i a_{in}(t)\ell_i(t)=1 \quad (3.20)$$

$\ell_i(t)$ — the vertically hyper-integrated labour coefficient for sector $i$ — is the sum of three components: $a_{ni}(t)$, i.e. direct labour for the production of one unit of final consumption commodity $i$ (direct labour); $T_i^{-1}a_{nk_i}(t)$, i.e. direct labour for the replacement of worn-out units of productive capacity for vertically hyper-integrated sector $i$ (indirect labour); and $(g+r_i)a_{nk_i}(t)$, i.e. direct labour required for the expansion of productive capacity of sector $i$ according to the growth of final demand for consumption good $i$ (hyper-indirect labour).[12]

We can now immediately take advantage of the just given definition in order to express prices in terms of vertically hyper-integrated labour. When conditions (3.15) hold, the prices of final consumption commodities in (3.5) can be written as:

$$p_i(t)=\left(a_{ni}(t)+\frac{1}{T_i}a_{nk_i}(t)+(g+r_i)a_{nk_i}(t)\right)\overline{w}+(\pi_i(t)-(g+r_i))p_{k_i}(t)$$

or

$$p_i(t)=\ell_i(t)\overline{w}+(\pi_i(t)-(g+r_i))p_{k_i}(t) \quad (3.21)$$

for $\quad i=1,2,\ldots,m$

It is interesting to notice that expression (3.21) establishes the production price of each final consumption commodity $i$ as the sum of two components: the cost of vertically hyper-integrated labour embodied in it — $\ell_i(t)\overline{w}$ — and a profit-differential with respect to the sectoral equilibrium rate of new investment — $(\pi_i(t)-(g+r_i))$ — computed on the value of equilibrium productive capacity at current production prices — $p_{k_i}(t)$. This second component is not the (dual) value counterpart of necessary physical quantity requirements of (re-)production and expansion, but emerges as an amount of purchasing power created in excess to these requirements, that goes into the owners of the means of production through

[12] For details, see Pasinetti (1981, p. 102).

the process of income distribution. The amount of this magnitude is a direct consequence of the theory of the income distribution that shall be adopted to close the price system, and it will influence the whole process of structural dynamics, via its effect on the pattern of expenditure of real income.

Another important magnitude which we shall introduce into the analysis is the level of equilibrium employment in each sector $i$, given by the product of the corresponding vertically hyper-integrated labour coefficient and the physical quantity of final consumption commodity $i$ produced during the time period:

$$L_i(t) = \ell_i(t)x_i(t), \qquad i = 1, 2, \ldots, m \tag{3.22}$$

It is relevant to stress the vertically hyper-integrated character of $L_i(t)$: in the most general specification of technology, a fraction of the total labour employed by a single industry producing a *basic* commodity would enter into the employment of *all* vertically hyper-integrated sectors, either directly and/or (hyper-)indirectly.

The comparison with the vertically hyper-integrated nature of the sectoral capital/output ratios — $\chi_i(t)$ in (3.16) — is straightforward. The composition of sectoral employment reflects not only the change in labour requirements of the industry producing the final consumption commodity concerned, but also the changing physical composition of the corresponding unit of productive capacity, and therefore the changes in labour requirements of *all* the industries composing the sector. It is for this reason that evaluating only the change in direct labour requirements cannot account for the interdependent and systemic nature of productivity changes. This opens up the possibility of performing empirical investigations on the dynamics of productivity taking vertically hyper-integrated sectors as the unit of analysis.[13]

We can now specify the dynamics of $\ell_i(t)$. By defining, for any variable $y(t)$ in the system, $\dot{y}(t) \equiv dy(t)/dt$, we can write:

$$-\frac{\dot{\ell}_i(t)}{\ell_i(t)} \equiv \varrho_i'(t) = \varrho_i \frac{a_{ni}(t)}{\ell_i(t)} + \varrho_{k_i} \frac{T_i^{-1} a_{nk_i}(t)}{\ell_i(t)} + \varrho_{k_i} \frac{(g + r_i) a_{nk_i}(t)}{\ell_i(t)} \tag{3.23}$$

$\varrho_i'(t)$ is the rate of growth of vertically hyper-integrated labour productivity of sector $i$, given by the weighted average of the rates of growth of direct, indirect and hyper-indirect labour productivity, the weights being the proportions of the three kinds of labour to total labour employed in vertically hyper-integrated sector $i$, respectively.

[13]For an empirical study taking this direction, see Garbellini & Wirkierman (2010a).

## 3.7 Dynamic equilibrium path for relative quantities, sectoral employment and relative prices

At this point, it is possible to describe the equilibrium path of relative quantities, sectoral employment and relative prices.

Let us start from the general dynamic movements for relative quantities and prices — given by expressions (3.8) and (3.9). As we have assumed that the capital accumulation equilibrium conditions (3.15), as well as the effective demand condition (3.20), hold, we can now specify the equilibrium path of relative quantities and prices, and the evolution of sectoral employment — given by expressions (3.22).

If "we choose to reckon prices in terms of Classical 'labour commanded' " (Pasinetti 1981, p. 99), the wage rate still being the basis for the price system, we set $\overline{w} = 1$. Hence, the equilibrium dynamic path of relative physical quantities, sectoral employment and commodity prices is given by:[14]

$$\begin{cases} x_i(t) = a_{in}(0)\overline{x}_n(0)e^{(g+r_i)t} \\ x_{k_i}(t) = \left(\frac{1}{T_i} + g + r_i\right) a_{in}(0)\overline{x}_n(0)e^{(g+r_i)t} \end{cases} \tag{3.24}$$

$$\Big\{ L_i(t) = \ell_i(t)a_{in}(0)\overline{x}_n(0)e^{(g+r_i)t} \tag{3.25}$$

and

$$\begin{cases} p_i^{(w)}(t) = \ell_i(t) + (\pi_i(t) - (g+r_i))\, a_{nk_i}(0)e^{-\varrho_{k_i}t} \\ p_{k_i}^{(w)}(t) = a_{nk_i}(0)e^{-\varrho_{k_i}t} \end{cases} \tag{3.26}$$

for $\quad i = 1, 2, \ldots, m$

where the expression for $p_i^{(w)}(t)$ is already written in the same form as in (3.21).[15]

The equilibrium solutions for physical quantities, given by expressions (3.24), together with equilibrium sectoral employment, given by expressions (3.25), represent a set of growing subsystems, one for each final consumption commodity $i$. Each growing subsystem or, equivalently, hyper-subsystem consists of three components: $x_i(t)$, $x_{k_i}(t)$ and $L_i(t)$. The first one represents the "production of one single consumption good $i$, expanding through time at its particular rate of growth $(g + r_i)$" (Pasinetti 1988, p. 127). The second one, represents the physical

[14] For a complete analysis of the equilibrium structural dynamics of a growing economic system, see Pasinetti (1981, pp. 91-99).

[15] In what follows, whenever a *nominal* magnitude has a letter in brackets as a superscript, that letter will indicate the *numéraire* commodity adopted. Therefore, $p_i^{(w)}(t)$ indicates the price of commodity $i$ when the *numéraire* of the price system is the wage rate.

quantities for "the maintenance of a circular production process that *both* reproduces all the means of production which are absorbed by the production process for [each] consumption good [...] *and* also produces those means of production that are strictly necessary to expand such a circular process at a rate of growth $(g+r_i)$" (Pasinetti 1988, p. 127). Finally, the third one represents the "absorption of a physical quantity of labour $L_i(t)$" (Pasinetti 1988, p. 127) required to produce physical quantities $x_i(t)$ and $x_{k_i}(t)$.

To see the implied structural dynamics, it is worth computing the rates of change of relative quantities, sectoral employment and relative prices:

$$\frac{\dot{x}_i(t)}{x_i(t)} = \frac{\dot{x}_{k_i}(t)}{x_{k_i}(t)} = g + r_i \tag{3.27}$$

$$\frac{\dot{L}_i(t)}{L_i(t)} = g + r_i - \varrho_i'(t) \tag{3.28}$$

$$\frac{\dot{p}_i^{(w)}(t)}{p_i^{(w)}(t)} \equiv \sigma_i^{(w)}(t) = -\varrho_i'(t)\frac{\ell_i(t)}{p_i^{(w)}(t)} + \left(\frac{\dot{\pi}_i(t)}{\pi_i(t)-(g+r_i)} - \varrho_{k_i}\right)\frac{(\pi_i(t)-(g+r_i))p_{k_i}^{(w)}(t)}{p_i^{(w)}(t)} \tag{3.29}$$

$$\frac{\dot{p}_{k_i}^{(w)}(t)}{p_{k_i}^{(w)}(t)} \equiv \sigma_{k_i}^{(w)}(t) = -\varrho_{k_i} \tag{3.30}$$

where $\sigma_i^{(w)}(t)$ is the rate of change of the relative price of commodity $i$ when the *numéraire* of the price system is the wage rate.[16]

As regards sectoral physical quantities, their equilibrium evolution, given by expressions (3.27), is completely determined by the evolution of effective demand for the corresponding final consumption commodity $i$ on which each growing subsystem is built. This holds true for both $x_i$ and $x_{k_i}$, due to the adoption of the units of productive capacity as the particular units of measurement for capital goods. Hence, the rate of change of physical quantities is given by the sum of two components: the rate of growth of population, $g$, common to all sectors; and the rate of change of sectoral per-capita demands for final consumption commodities, $r_i$, specific to each sector. Since these are different from sector to sector, the whole structure of relative quantities is changing through time.

As regards sectoral employment, its equilibrium evolution, given by expressions (3.28), is determined both by the equilibrium evolution of relative quantities, and by the dynamics of vertically hyper-integrated labour productivities. Since $r_i$ is different from $\varrho_i'(t)$, and both are different from sector to sector, the whole

[16] It must be carefully noticed that expressions (3.29) have a finite value for $\pi_i(t) \neq g + r_i, \quad i = 1, 2, \ldots, m$.

structure of employment, i.e. the division of labour within the economic system, is continuously changing through time. This makes clear how a *sectoral reallocation* of employment is an essential requirement for the system to follow a full-employment path. It is worth noticing that, at the most general description of technology, a change in labour productivity in a single *industry* producing a basic commodity would be enough to change the *whole* structure of *sectoral* employment.

As regards relative prices, let us first notice that the equilibrium dynamics for the price of a unit of productive capacity for final consumption commodity $i$, given by expressions (3.30), is a particularly simple one, due to the simplifying assumption made, i.e. capital goods are produced by means of labour alone. As a consequence, prices are completely determined by labour costs, and therefore their equilibrium evolution only depends on the changes of labour productivity in the industry producing the specific capital good for sector $i$.

The equilibrium dynamics of commodity prices, given by expressions (3.29), reveals the process of change of a production price brought about by the interaction of technical progress and changes in the distribution of income. The rates of change of commodity prices are given by the weighted average of the rates of change of their two components. The first addendum shows how an increase in vertically hyper-integrated labour productivity exerts a univocally negative effect on production prices. The second addendum quantifies the effect of a change in income distribution — through a variation in the sectoral profit rate — on the 'labour commanded' production prices.

As we have already said, the first component reflects a necessary, physical self-replacement and expansion requirement; accordingly, its rate of change is completely determined by technology and equilibrium new investment, i.e. by the rate of change of vertically hyper-integrated labour $\varrho_i'(t)$. On the contrary, the second component of production price also reflects income distribution. Accordingly, its rate of change depends not only on labour productivity in the capital goods producing industry — i.e. the rate of change of the price of the unit of productive capacity on which profits are computed — but also on the variation through time of the sectoral rate of profit — i.e. on the rate of change of the profit differential with respect to the sectoral equilibrium rate of new investment.

This analysis completes the description of the structural equilibrium dynamics of a growing economic system. In fact, we have described *a set* of equilibrium paths, one for each possible realisation of the *sequence* of sectoral rates of profit, so far considered as exogenous magnitudes.

# 4 The 'natural' economic system

We are now in a position to introduce what Pasinetti calls the 'natural' economic system, i.e. that particular equilibrium path associated to one specific sequence of sectoral rates of profit, that, "without recourse any longer to any exogenously given economic magnitude, now come to complete and close the whole relative price system of our theoretical scheme" (Pasinetti 1981, p. 131), due to the adoption of a particular *theory of the rate of profit*.

## 4.1 The 'natural rates of profit'

As we have explained in section 2, the aim of Pasinetti's (1981) book is that of developing a framework explaining the 'primary and natural' features of a growing economic system, independently of a particular institutional set-up. A reasoning in these terms, when coming to the issue of the distribution of income, would seem, at first sight, counterintuitive, since the way in which income is distributed crucially depends on the character of the *social relations of production*, no less than on cultural, ethic, legal considerations, that is to say, precisely on the institutional set-up of society. In fact, those analyses taking income distribution as exogenous, are clearly embedded in a specific institutional set-up.

Then, how can a theory of the rate of profit be conceived that is independent of it?

As Pasinetti states, the 'natural' economic system deals with *logical* relations, based on magnitudes given from outside economic analysis (and therefore taken as exogenous), and emerging from the physical growth requirements of the system itself. The problem must be therefore faced from this perspective: is there "a natural rate of profit (...) already *logically* implied in the previous theoretical framework *because the economic system* considered *is a growing one*"? (Pasinetti 1981, p. 128, italics added)

The answer to this question is: yes.

The crucial point is that at a pre-institutional stage of the analysis, a theory of the rate of profit is *not* a theory of income distribution among income recipients, i.e. individuals or groups of individuals. This is because the very definition of the categories among which the purchasing power generated in the process of production is to be distributed essentially depends on the social relations of production of a particular institutional set-up.

However, the very nature of an industrial system requires to perform a separation between the means of production that enter a circular process, and the set of commodities that are left out from the circular flow, once they are produced. Moreover, when the system is a growing one, the new investment requirements become a *necessary* expansion of the means of production.

Now, hence, prices of production must, on the one hand, be precisely those exchange ratios that satisfy the conditions of re-production and growth — i.e. that *include* the growth of the means of production at the equilibrium rate of accumulation. But given that the equilibrium requirement to expand productive capacity differs among vertically hyper-integrated sectors, the surplus factor in the price of production of each consumption commodity must reflect this difference.

On the other hand, prices of production provide for the purchasing power both to self-replace and expand productive capacity and to consume those commodities not re-entering the circular flow. Consider that profits and wages just establish the amount of purchasing power that must be channeled to demand for means of production to expand productive capacity and to demand for final consumption commodities, respectively. In this sense, profits and wages would establish a truly *functional* distribution of income, as they stand for categories that channel purchasing power to two different economic *functions*. These functions arise from the conditions of production of physical quantities, in particular, from the need to separate what enters the circular flow (and is used as means of production) from what it does not (and is consumed).

As a consequence, from the reasoning stated above, it follows that profits must correspond to the purchasing power necessary for the equilibrium expansion of productive capacity in each vertically hyper-integrated sector to take place. In formal terms:

$$\pi_i^*(t)p_{k_i}(t)k_i(t) = (g + r_i)p_{k_i}(t)x_i(t), \quad \forall i = 1, 2, \ldots, m; \quad t \geq 0 \tag{4.1}$$

and therefore, since in equilibrium $x_i(t) = k_i(t)$:

$$\pi_i^*(t) = \pi_i^* = g + r_i, \quad i = 1, 2, \ldots, m \tag{4.2}$$

The equilibrium configuration corresponding to the just given structure of rates of profit is the only one that keeps the analysis at a strictly pre-institutional level.

## 4.2 A pure labour theory of value

The rates of profit in (4.2) are the 'natural' rates of profit. When inserted into the equilibrium solutions for consumption commodity prices in (3.26), these become:

$$p_i^{(w)*}(t) = \ell_i(t), \quad i = 1, 2, \ldots, m \tag{4.3}$$

their rate of change through time being:

$$\frac{\dot{p}_i^{(w)*}(t)}{p_i^{(w)*}(t)} = -\varrho_i'(t), \quad i = 1, 2, \ldots, m \tag{4.4}$$

Expressions (4.3) highlight the main result of the present formulation: when labour is the *numéraire* of the price system, and the rates of profit are the natural ones, prices — i.e. 'labour commanded' prices — come to be exactly equal to 'labour embodied'. Therefore, this theoretical scheme implies a generalisation of a *pure labour theory of value*, where the equality of 'labour commanded' and 'labour embodied' is achieved thanks to a "re-definition of the concept of 'labour embodied', which must be intended as the quantity of labour required directly, indirectly and hyper-indirectly to obtain the corresponding commodity as a consumption good" (Pasinetti 1988, pp. 131-132).

With the introduction of the 'natural' rates of profit, both the value of productive capacity for self-replacement and the profits computed on the value of existing productive capacity have the same function of "computing amounts of labour indirectly required elsewhere in the economic system for the equilibrium production of consumption good $i$" (Pasinetti 1981, p. 132).

Not less importantly, this result holds not at the level of the economic system as a whole, but in each single sector. Each growing subsystem following an equilibrium path of accumulation has a 'dual' value side that ascribes to natural prices a straightforward foundation, based on the "basic principle of equal rewards for equal amounts of homogeneous labour." (Pasinetti 1981, p. 133).

Equally interesting, at the most general specification of technology, the labour embodied in the basic commodities produced by a single industry participate in profits of all vertically hyper-integrated sectors. In this way, changes in the productivity of *labour* in an industry alter the value of *profits* on capital of all sectors. Thus, it becomes clear that "it is not the 'productivity of capital', or of any commodity, that turns out to be the *raison d'être* of the rate of profit. It is the growth, and the increasing productivity, of labour!" (Pasinetti 1981, p. 133).

## 4.3 Natural profits, wages, new investments and consumption

There is a very clear asymmetrical relation between total natural profits and wages. The total national income produced in a specific time period, i.e. the *value* of total production, net of replacements, at current prices, is distributed among total (natural) profits and total wages. While the former emerge from the *physical* conditions for equilibrium growth as a necessity, if full employment and full capacity utilisation are to be maintained through time, the latter can be seen as a 'surplus', absorbing all the remaining national income. "To produce, and to continually increase this 'surplus', through technical progress, is precisely the purpose of the whole production process" (Pasinetti 1981, p. 144).

In the same way, there is an asymmetric relation between total new investments and consumption. The total quantities produced in a specific time period, net of replacements, must be devoted in part to new investments and in part to final consumption. While the former are determined — by the structure of final demand for consumption and its evolution through time — as a *physical* requirement for equilibrium growth, the aggregate level of the latter can be seen as a 'surplus', absorbing all the remaining purchasing power. Again, "to produce this surplus, and to continually increase it through technical progress, is the whole purpose of the production process" (Pasinetti 1981, p. 146).

As can be seen by the very definition of natural profits, emerging from conditions (4.1), in the 'natural' economic system, total profits will be equal to the value, at current prices, of total new investments, and correspondingly total wages will be equal to the value of total final consumption. But what is even more interesting is that this holds not as "a mere over-all averaging-out result, but [as] the consequence of a whole series of equalities realised at each single sectoral stage" (Pasinetti 1981, p. 147). In fact, condition (4.1) establishes that:

$$p^*_{k_i}(t)(g + r_i)a_{in}(t)x_n(t) = p^*_{k_i}(t)\pi^*_i x_i(t), \quad i = 1, 2, \ldots, m \tag{4.5}$$

and from the expression for natural prices of consumption commodities (4.3) it follows that:

$$p^*_i(t)a_{in}(t)x_n(t) = w(t)L_i(t), \quad i = 1, 2, \ldots, m \tag{4.6}$$

As a consequence, the value, at current prices, of total quantities, net of replacements, produced in each vertically hyper-integrated sector equals the total income it generates, i.e.:

$$p^*_i(t)a_{in}(t)x_n(t) + p^*_{k_i}(t)(g + r_i)a_{in}(t)x_n(t) = w(t)L_i(t) + p^*_{k_i}(t)\pi^*_i x_i(t) \tag{4.7}$$
$$\text{for} \quad i = 1, 2, \ldots, m$$

## 4.4 Changes in productivity and distributive variables

A straightforward consequence of expressions (4.4) is that any price reduction due to increases in labour productivity immediately translates into a corresponding increase in the real purchasing power of wages. This can be seen even more clearly by changing the *numéraire*.

Any commodity or composite commodity can be chosen as the *numéraire* of the price system; analytically, this amounts to setting its price equal to unity, and keeping it constant through time. For example, if commodity $h$ is chosen as the *numéraire*, we set:

$$\begin{cases} p^{(h)}_h(0) = 1 \\ \sigma^{(h)}_h(t) = \sigma^{(h)}_h = 0 \end{cases} \tag{4.8}$$

Once the *numéraire* is specified, the wage rate has to be expressed in terms of it; this again means closing *two* degrees of freedom, i.e. we have to set both the wage rate at time zero and its rate of change in terms of the chosen *numéraire*. Within the ‘natural’ economic system, again taking commodity $h$ as the *numéraire*, this means setting:

$$p_h^{(h)*}(t) = w(t)\ell_h(t) = 1 \tag{4.9}$$

from where we obtain:

$$w^{(h)}(t) = (\ell_h(t))^{-1} \tag{4.10}$$

and, therefore, we set:

$$\begin{cases} w^{(h)}(0) = (\ell_h(0))^{-1} \\ \frac{\dot{w}^{(h)}(t)}{w^{(h)}(t)} \equiv \sigma_w^{(h)}(t) = \varrho'_h(t) \end{cases} \tag{4.11}$$

and the rate of change of the price of any consumption commodity $i$ is given by:

$$\frac{\dot{p}_i^{(h)*}(t)}{p_i^{(h)*}(t)} = \varrho'_h(t) - \varrho'_i(t) \tag{4.12}$$

Hence, the rate of change of the wage rate in terms of the chosen *numéraire* — the *real* wage rate — is given by the rate of increase of labour productivity in the corresponding vertically hyper-integrated sector, and the rate of change of the price of commodity $i$ in terms of the chosen *numéraire* is given by the difference of the rate of change of labour productivity in the corresponding sector with respect to the rate of change in vertically hyper-integrated labour productivity in the sector producing the *numéraire* commodity.

As a consequence, within the ‘natural’ economic system, the dynamics of the wage rate and the sectoral rates of profit have two different orders or magnitude (see Pasinetti 1981, p. 143). The *level* of each $\pi_i^*$ in (4.2) is given by two constant rates of change,[17] while the *rate of change* of the *real* wage rate is given by the rate of change of labour productivity in the vertically hyper-integrated sector producing the commodity chosen as *numéraire*. “In the long run, therefore, while the real wage rate will persistently grow, the rate(s) of profit cannot but roughly remain at the same level” (Pasinetti 1981, p. 143).

[17]When the hypothesis of steady rate of change of per capita demand for consumption commodity $i$ ($i = 1, 2, \ldots, m$) is removed, the natural rates of profit are no more exactly constant through time, but shall exhibit a roughly constant trend.

### 4.5 Natural structural dynamics

By closing the relative price system with the specific structure of sectoral rates of profit given by expressions (4.2) — the 'natural' rates of profit — we are actually closing the last degree of freedom left open at the end of section 3, focusing on the particular dynamic equilibrium path of:

(i) relative physical quantities of each growing subsystem — expressions (3.24) and (3.27);
(ii) sectoral employment — expressions (3.25) and (3.28);
(iii) final consumption commodity (relative) prices — expressions (4.3) and (4.4);
(iv) (relative) prices of the units of vertically hyper-integrated productive capacity — the second series of expressions in (3.26) and expressions (3.30).

which constitute the complete description of the 'natural' economic system.

It is worth concluding our description of the 'natural' economic system by noticing that the only explicitly different analytical formulations, with respect to the general case, are given by expressions (4.3) and (4.4), concerning final consumption commodity prices. The relative physical quantity system and sectoral employment would apparently be the same, irrespective of the particular rates of profit chosen.

However, this result is a consequence of the fact that, in this framework, demand coefficients are taken as given. But the structure of demand is strongly dependent on consumers' real income, which in turn is determined by the structure of relative prices and therefore also by the ruling rate(s) of profit. To be more precise, therefore, we are not considering the $a_{in}(t)$'s as exogenous, but we are considering as exogenous the *mechanism* by which changes in income distribution modify the structure of final consumption demand (and therefore of relative physical quantities). Such mechanism is constantly at work.[18] Demand coefficients in expressions (3.24) and (3.25) may therefore be different according to the particular configuration of the rate(s) of profit.

## 5 A methodological note

It might be useful to open this brief note on Pasinetti's (1981) method by recalling the definition of equilibrium, already given at the beginning of section 3, adopted all throughout the book:

> A *situation of equilibrium* will simply be taken to mean a situation in which there is full employment of the labour force and full utilisation of the existing productive capacity.

[18] See Pasinetti (1981, pp. 71-77).

(Pasinetti 1981, pp. 48-49)

It is not trivial that Pasinetti is referring to a '*situation* of equilibrium'; the choice of the term highlights the transitory character of any equilibrium position eventually reached at a certain point in time. In fact, "no connotation of automatism and no association with any particular adjustment mechanism is intended to be implied by such an expression" (Pasinetti 1981, p. 48).

The 'natural' economic system is by no means an attempt at describing the functioning of an actual capitalist system; it is an attempt at singling out the 'primary and natural' features of an *industrial* system intended, i.e. "necessary requirements for equilibrium growth" (Pasinetti 1981, p. 25). Equilibrium growth, however, entails neither the identification of a 'normal position' towards which the system tends in the long run — since the very structural dynamics of the economic system makes it impossible to identify a 'normal position' *persistent* enough to the continuous changes in the system's proportions — nor a logical *succession* of temporary equilibria spontaneously realised.

The equilibrium dynamics defining the 'natural' economic system specifies the re-proportioning of productive capacity, relative quantities — and therefore sectoral employment — and relative production prices *necessary to comply with* the ever-changing structure of final demand for consumption goods and with the pace of technical progress. It is important to stress that this process of re-proportioning is not *spontaneous*, but must be actively pursued if the new situation of equilibrium is to be reached period after period.

Furthermore, the empirical point of departure of the analysis must be explicitly mentioned: "The coefficients that appear [...] in the present (vertically [hyper-]integrated) analysis must [...] be interpreted as representing those physical quantities which can actually be observed" (Pasinetti 1981, p. 110), to which there corresponds — for *each* time period — a specific equilibrium situation. These equilibrium situations, together with the *necessary* dynamic conditions connecting them through time, establish a 'normative configuration'. In this sense, therefore, the 'natural' economic system is a "norm; and the norm is always there — even if it is not so much apparent — in the *short* no less than in the long run" (Pasinetti 1981, p. 127n).

Since the whole structure of physical quantities and technical production requirements are continuously changing through time, the problem arises of how to perform a truly dynamic analysis, connecting equilibrium situations with completely different characteristics. Pasinetti solves the problem by developing the analytical device of vertical hyper-integration: "By resolving all varieties of products into the same constituent elements — a flow of labour and a stock of capital goods both expressed in physical terms — the vertically [hyper-]integrated approach leads to relations whose permanence over time is independent of specific technical possibilities" (Pasinetti 1981, p. 116).

It worth stressing, however, that vertical hyper-integration is not a mere analytical device for making dynamic analysis possible; it also has a very important conceptual role within the development of the present theoretical framework — consider, for example, its role in the redefinition of the concept of 'labour embodied' in the theory of value implied by the 'natural' economic system.

To conclude, the 'natural' economic system is strongly rooted on the notion of vertically hyper-integration. The stages of development of this concept and its connection to Pasinetti's (1981) model of *Structural Change and Economic Growth* are explored in the next section.

# 6 Development of the concept of vertical hyper-integration

The concept of vertical hyper-integration, together with its analytical formulation, is one of the cornerstones of the approach to structural dynamics put forward by Pasinetti. However, its development has not been immediate, as it clearly went through different stages. Pasinetti's (1981) book has been the final result of a process that began with his Doctoral dissertation at the University of Cambridge (Pasinetti 1962a) — partially published in Pasinetti (1965). However, the book has itself been an intermediate outcome as regards the analytical elaboration of the device of vertical hyper-integration, which came to final accomplishment in Pasinetti (1988).

In his first general treatment of vertical integration (Pasinetti, 1973), the author explicitly recognised that he has "always been faced with questions" (Pasinetti 1973, p. 2) on "how to construct the vertically integrated sectors in the general case" (Pasinetti 1973, p. 2n), i.e. beyond the particular case where capital goods are made by labour alone. In fact, indications for proceeding towards a generalisation, in Pasinetti (1965), had been given "in a brief and incomplete way" (Pasinetti 1973, p. 2).

In Pasinetti (1973, p. 2) "The economic system is supposed to be viable, in the sense that is capable of producing larger quantities of commodities than those required to replace used-up capital goods". From this basic assumption on the character of the circular process, the analysis proceeded towards obtaining a compact way to represent sub-systems in the sense of Sraffa (1960). In this case, the net output consisted of both final consumption commodities and new investment goods. However, the crucial difference between vertical integration and vertical hyper-integration departs from the consideration of new investments as belonging either to the net output — and therefore not entering the circular process — or to the means of production which re-enter the circular flow. As stated clearly in Pasinetti (1988):

> We now proceed in a way perfectly analogous to the one used in defining the earlier subsystems in Pasinetti (1973), but with the essential difference here of including in each hyper-subsystem *all* gross investments (both replacements *and* net investments).
>
> (Pasinetti 1988, p. 127)

In Pasinetti (1973), the analytical focus was mainly on the role of vertically integrated sectors in the theory of value and income distribution, while the analysis of economic growth with technical progress has been done by recalling the particular case of capital goods produced by labour alone. In fact, it has been only within the 'dual' exercise of an economy growing at a steady rate $g$ in absence of technical progress that new investments were explicitly treated as part of the means of production expanding at the uniform growth rate. And the exercise was performed exclusively "in the search for an equilibrium growth solution" (Pasinetti 1973, p. 20).

In this sense, the analysis of dynamic models of equilibrium growth has proven to be an important intermediate analytical step in the discovery of vertical hyper-integration. It has been precisely within the discussion of the full-employment condition of the Dynamic Input-Output model in Pasinetti (1977, Chapter VII) that the first explicit reflection on hyper-indirect requirements came about:

> For every given exponential evolution of consumption [...] the solution [...] gives the evolution of the total physical quantities $\mathbf{Q}(t)$ which are required — as direct, indirect, and, we might add, hyper-indirect requirements (meaning by the last the requirements for new investment) — to keep the economic system in dynamic equilibrium.
>
> (Pasinetti 1977, p. 196)

It is interesting to note that the concept is formulated within the discussion of the necessary conditions for dynamic solutions to comply with full capacity utilisation and full employment of the labour force. The author reaches the conclusion that if the initial situation of the system is an appropriate one, both full employment and full capacity utilisation would follow through time.

The analysis of the Dynamic Input-Output model in Pasinetti (1977) marked a sharp difference with respect to Pasinetti (1973). The focus was not on the analytical treatment of subsystems but on the derivation of dynamic equilibrium solutions, on the one hand highlighting the method of analysis — by which the level (and structure) of per capita consumption was the only component of net-output considered as given (Pasinetti 1977, pp. 194-195) — and, on the other hand, showing the restrictions on the choice of the consumption structure imposed by the maximum rate of growth (Pasinetti 1977, p. 209). This second aspect would be taken up again in the Appendix to Chapter VI of Pasinetti (1981), where important insights have been presented on the difficulties of relying on von Neumann proportional dynamics to analyse systems undergoing technical change.

But then, how come the analysis of the Dynamic Input-Output model had already been carried out with an explicit identification of vertically hyper-integrated magnitudes, while Pasinetti (1981) made only partial treatment of vertical hyper-integration? Hints at the stages of development of Pasinetti (1981) are found in the preface of the book. At this juncture — as has been pointed out in section 3.2 — Pasinetti himself had already begun to think in terms of vertically hyper-integrated sectors, though this is only reflected in some parts of the book. In fact, it can be seen that (almost) all the entries in the index concerning vertical hyper-integration belong to the chapters of the book which "have been almost entirely re-written" (Pasinetti 1981, p. xiv) since the time of his PhD Thesis, while in the remaining parts of the book expressions referring to vertically integrated magnitudes are still present.

It is clear from the analysis carried out so far that the model in Pasinetti (1981) adopts a method of singling out conditions for dynamic equilibrium based on necessary physical requirements for self-replacement and expansion — therefore considering new investments as part of the means of production re-entering the circular flow — but instead of dealing with a system growing at a uniform rate, develops the analysis within the framework of growing subsystems, even though a very simplified description of the technique is adopted — the one already present in Pasinetti (1965).

Therefore, it is our contention that Pasinetti's (1981) *Structural Change and Economic Growth* presents a vertically hyper-integrated model, though within a very simplified description of the technique. Many insights are further enriched when the model is seen through these lens. For example, the role of the vertically hyper-integrated units of productive capacity in the analysis of accumulation (presented in section 3.2) and that of vertically hyper-integrated labour coefficients in the analysis of value (presented in section 3.6 and in section 4.2 within the 'natural' economic system). Of course, a full generalisation of vertical hyper-integration in the context of non-proportional growth would only arrive with Pasinetti (1988), where growing subsystems would acquire their most general formulation.

# Structural Change and Economic Growth: Production in the Short Run — A generalisation in terms of vertically hyper-integrated sectors

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**Abstract** Pasinetti's (1981) *Structural Change and Economic Growth* provides a complete and far reaching theoretical framework for the study of structural change, and therefore of economic development, rooted in in the Classical-Sraffian tradition.

Some attempts have been made, both in the '80s — for instance Siniscalco (1982) and Momigliano & Siniscalco (1986) — and more recently — e.g. Montresor & Vittucci Marzetti (2007a) and Montresor & Vittucci Marzetti (2008) — to use this framework for empirical purposes. However, all these attempts are based on Pasinetti's (1973) paper, i.e. on vertically integrated analysis. It is my contention that, as a consequence, they failed to recognise, and therefore to take advantage of, the main analytical feature of the 1981 book, namely vertical *hyper-integration*.

Actually, when trying to overcome the simplifying assumptions made by Pasinetti (1981) as regards the description of the technique, the starting point should be Pasinetti (1988), and not Pasinetti (1973), the latter being an intermediate step leading to the former.

The aim of the present paper is therefore, first of all, that of highlighting the key differences between Pasinetti (1973) and Pasinetti (1988), in order to show Pasinetti's (1981) vertically *hyper*-integrated character.

In the second place, the whole analytical framework provided by Pasinetti (1981) will be generalised by reintroducing inter-industry relations and allowing for more complex dynamics of economic magnitudes.

This conceptual clarification and analytical generalisation is intended to be the first step of a line of research aiming at using, and extending, the present framework to perform empirical analyses and study the behaviour of actual economic systems.

**Keywords** Natural system, vertically integrated sectors, vertically hyper-integrated sectors, functional income distribution, natural rates of profit, natural prices.

**JEL classification** B51,L16,O41

# 1 Introduction

Pasinetti started developing his multi-sectoral framework at the beginning of the Sixties, with his doctoral dissertation (see Pasinetti 1962a). The development of such a framework went through different stages,[1] the milestones of which are Pasinetti (1973), Pasinetti (1981) and Pasinetti (1988).

If the latter work has provided us with a full and explicit generalisation of the notion of vertically integrated sector — namely, with the introduction of the concept of vertically *hyper*-integrated sector, or growing *subsystem* — Pasinetti's (1981) book, though being *naive* in some analytical respects (the very notion of vertically hyper-integrated sector was already *in pectore*, but not completely elaborated), touches upon a great deal of theoretical and practical issues, giving us a reading key to face many problems which have been left unsolved by former economic theory, and most of all many insights to go on working with the Classical/Sraffian approach, by overcoming its major shortcoming — the difficulty in dealing with dynamics and hence with *growth*, which is, without any doubt, the most important feature of all modern economic systems.

It is my contention, therefore, that such an approach to economic theory is a very important starting point to go "back to the future"[2] of Classical Political Economy.

In order to fruitfully do so, however, some preliminary work needs to be done, mainly to fill the gap between Pasinetti (1981) and Pasinetti (1988). This paper is intended to be one of the necessary building blocks.

After presenting, in section 2, the basic notation that will be used all throughout the paper, section 3 provides a brief presentation of the traditional industry-level framework at the basis of Modern Classical Economics. Such a summary is intended to be a reference point to fully understand the main innovations introduced by Pasinetti's work.

Section 4 then presents the main features and categories of vertically integrated (Pasinetti 1973) and vertically *hyper*-integrated (Pasinetti 1988) analysis, trying to stress and clarify the differences between the two, with particular attention to

[1] For details on the stages of development of the concept of vertically hyper-integrated sectors, see Garbellini & Wirkierman (2010b, section 6).

[2] To cite Pasinetti himself: Pasinetti (2007, p. 329).

the way in which new investment is treated and therefore net output is defined.

Section 5 then goes to *Structural Change and Economic Growth*, and is divided into three subsections.

Section 5.1 presents the original formulation, though restated in matrix terms and solved as an eigenproblem.

Sections 5.2 and 5.3 are attempts at taking the frameworks developed, respectively, by Pasinetti (1973) and Pasinetti (1988) and restating them in terms analogous to those of Pasinetti (1981), introducing the same categories, magnitudes, and equilibrium conditions — first in vertically integrated and then in vertically *hyper*-integrated terms.

This restatement aims at making it clear that Pasinetti (1981) represents an intermediate stage towards the elaboration of the notion of *growing subsystems*, by stressing both the novelties with respect to Pasinetti (1973) and the analogies with Pasinetti (1988). At the same time, section 5.3 is intended to be the basis for further generalisation of Pasinetti's (1981) framework in vertically hyper-integrated terms and with a more realistic description of the technique in use.

Finally, section 6 is a note on the price system, section 7 discusses some relevant sectoral and aggregate economic magnitudes, and section 8 provides some final remarks.

The Appendices include some algebraic manipulations which I have left implicit in the paper not to take the reader's attention away from the development of the main arguments.

## 2 Basic notation

Consider an economic system in which $m$ commodities, denoted by the subscript $i$ ($i = 1, 2, \ldots, m$) are produced. Such commodities can be used *either* as (pure) consumption goods *and/or* as intermediate commodities.

Moreover, make the simplifying assumption that those commodities used as means of production are completely used up in each period, and therefore have to be replaced entirely.[3]

The economic system can be described by:

[3] No treatment of fixed capital is made here. This simplification is intended to be a first step to be followed by a complete treatment of this issue too. However, since extending the description of the technology in use introduces many complications, I have decided to limit myself, for the time being, to consider circulating capital only.

| | | | |
|---|---|---|---|
| $\mathbf{q}$ | $=$ | $[q_i]$: | vector of total quantities; |
| $\mathbf{x}$ | $=$ | $[x_i]$: | vector of per-capita (average) final demand for consumption goods; |
| $\mathbf{j}$ | $=$ | $[j_i]$: | vector of final per-capita (average) demand for investment goods; |
| $\mathbf{y}$ | $=$ | $[y_i]$: | vector of final per-capita (average) demand, with $y_i = x_i + j_i$, $i = 1, 2, \ldots, m$; |
| $\mathbf{A}$ | $=$ | $[a_{ij}]$: | matrix of inter-industry coefficients; |
| $\mathbf{a}_{ni}$ | $=$ | $[a_{ni}]$: | vector of direct labour requirements; |
| $\mathbf{a}_{in}$ | $=$ | $[a_{in}]$: | vector of demand coefficients for consumption goods: $x_i = a_{in} x_n$; |
| $\mathbf{a}_{k_i n}$ | $=$ | $[a_{k_i n}]$: | vector of demand coefficients for new investment: $j_i = a_{k_i n} x_n$; |
| $\mathbf{s}$ | $=$ | $[s_i]$: | vector of intermediate commodities necessary for the production of quantities $q_i$; |
| $\mathbf{p}$ | $=$ | $[p_i]$: | vector of commodity prices; |
| | | $x_n$: | total labour. |
| | | $g$: | rate of growth of population; |
| | | $r_i$: | rate of growth of per-capita (average) demand of commodity $i$ as a final good; |
| | | | $(i = 1, \ldots, m)$ |

All throughout the paper, the following conventions will be observed:

- All vectors and matrices will be denoted by boldface symbols, while all scalar quantities by normal type ones;
- all matrices will be denoted by upper case letters, while all vectors by lower case ones;
- all vectors will be intended as column vectors; row vectors will be denoted by transposed vectors;
- a vector with a hat will denote a diagonal matrix with the element of the corresponding vector on the main diagonal.

# 3 Quantity and price system at the industry level

## 3.1 A stationary system

Let us suppose to start from a situation of *stationary equilibrium*, i.e. a situation in which the economic system produces, in each period, a total quantity of commodities equal to the final demand for consumption goods plus the productive capacity used up during the production process, in order to be able to satisfy, period after period, the same final demand for consumption goods.

Since there is no growth, there are no new investments, and therefore the net output is given only by final demand for consumption goods: $\mathbf{y} = \mathbf{x} = \mathbf{a}_{in} x_n$.

In such a case, the physical quantity system can be written as:

$$\mathbf{q} = \mathbf{A}\mathbf{q} + \mathbf{y} = \mathbf{A}\mathbf{q} + \mathbf{x} \tag{3.1}$$

and therefore:

$$\mathbf{q} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} \tag{3.2}$$

The physical quantities to be produced in the economic system as a whole are given by the direct and indirect physical requirements for the production of the goods entering the vector of final demand $\mathbf{x}$.

Since we aim at describing a situation of equilibrium,[4]we want labour force to be fully employed; we can therefore add a further equation, namely $x_n = \mathbf{a}_{ni}^T \mathbf{q}$, to the physical quantity system, which thus becomes:

$$\begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{a}_{in} \\ -\mathbf{a}_{ni}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \tag{3.3}$$

or, as an eigenproblem:[5]

$$\begin{cases} (\lambda_q \overline{\mathbf{I}} - \overline{\mathbf{A}}_q)\overline{\mathbf{q}} = \overline{\mathbf{0}} \\ \lambda_q^* = 1 \\ \lambda_q^* = \lambda_q^{max} \end{cases} \tag{3.5}$$

The solution vector, $\overline{\mathbf{q}}$, is the right-hand eigenvector of matrix $\overline{\mathbf{A}}_q$, associated with the eigenvalue $\lambda_q = \lambda_q^* = 1$ which, for $\overline{\mathbf{q}}$ to have all real and non-negative elements, must also be the maximum eigenvalue. In fact, since all elements of matrix $\overline{\mathbf{A}}_q$ are non-negative, we can exploit the Perron-Frobenius theorems, saying that the a non-negative matrix has only one non-negative eigenvector, i.e. the one associated to its maximum eigenvalue.[6]

[4]What the word ‘equilibrium’ means, in this context, has been already explored in Garbellini & Wirkierman (2010b). Suffice here to recall Pasinetti’s own words: a single period equilibrium is “a situation in which there is full employment of the labour force and full utilisation of the existing productive capacity” (Pasinetti 1981, pp. 48-49).

[5]Where:

$$\overline{\mathbf{A}}_q = \begin{bmatrix} \mathbf{A} & \mathbf{a}_{in} \\ \mathbf{a}_{ni}^T & 0 \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} \tag{3.4}$$

[6]For a synthetic exposition of Perron-Frobenius theorems for non-negative matrices, see Pasinetti (1977, pp. 267-276).

The characteristic polynomial associated to this eigenproblem is:

$$|\lambda_q \mathbf{I} - \mathbf{A}| \left(-\lambda_q + \mathbf{a}_{ni}^T (\lambda_q \mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in}\right) \tag{3.6}$$

In order to find the $2m + 1$ eigenvalues of matrix $\overline{\mathbf{A}}_q$, we have to find the solutions to the characteristic equation, i.e. those values of $\lambda_q$ making the characteristic polynomial equal to zero. The first factor of the polynomial, i.e. the determinant of matrix $(\lambda_q \mathbf{I} - \mathbf{A})$, cannot be zero, or the inverse $(\lambda_q \mathbf{I} - \mathbf{A})^{-1}$ would fail to exist.[7] Therefore, we concentrate our attention on the part in brackets:

$$\mathbf{a}_{ni}^T (\lambda_q \mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} - \lambda_q = 0 \tag{3.7}$$

We now want to find the conditions for $\lambda_q = \lambda_q^*$ to be an eigenvalue of matrix $\overline{\mathbf{A}}_q$, i.e. a solution of equation (3.7). In order to do so, we substitute $\lambda_q = \lambda_q^*$ into (3.7) itself:

$$\mathbf{a}_{ni}^T (\mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} = 1 \tag{3.8}$$

To see that matrix $\overline{\mathbf{A}}_q$ has no eigenvalues greater than $\lambda_q^*$, let us suppose that there exists an eigenvalue $\mu > 1$; this would imply that:

$$\mathbf{a}_{ni}^T (\mu \mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} = \mu \tag{3.9}$$

By Perron-Frobenius theorems, all elements of matrix $(\mu \mathbf{I} - \mathbf{A})^{-1}$ are decreasing functions of $\mu$; therefore, since $\mu > 1$, then $(\mu \mathbf{I} - \mathbf{A})^{-1} < (\mathbf{I} - \mathbf{A})^{-1}$, and hence:

$$\mathbf{a}_{ni}^T (\mu \mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} < 1 < \mu$$

which clearly leads to a contradiction.

Since $\lambda_q^* = \lambda_q^{max}$, and therefore the solution vector for physical quantities is real and non-negative for all possible vectors $\mathbf{a}_{ni}^T$ and $\mathbf{a}_{in}$, in order to completely determine it we have to fix arbitrarily one component, giving us the *scale* of the solution. For the physical quantity system case, the choice is quite obvious, since we have one magnitude — namely total population $x_n$ — which is determined outside the economic system, and which therefore can be taken as given. By setting $x_n = \overline{x}_n$, we can write the solution vector as:

$$\begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} = \begin{bmatrix} (\mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} \overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{3.10}$$

[7] This simply means that if matrix $\mathbf{A}$ has the same eigenvalues as matrix $\overline{\mathbf{A}}_q$, the inverse does not exist. As we will see later on, the maximum eigenvalue of matrix $\mathbf{A}$ must be smaller than one for gross quantities to be non-negative, while we will show that the maximum one of matrix $\overline{\mathbf{A}}_q$ is precisely one.

In conclusion, if $\lambda_q^*$ is the maximum eigenvalue of matrix $\overline{\mathbf{A}}$, and if condition (3.8) is satisfied, then $\overline{\mathbf{q}}$ is a vector of real and non-negative quantities[8], the solution to our eigenproblem.

Mathematically, expression (3.8) is a condition for our eigenproblem to have non-trivial solutions. From an economic point of view, it is a *macroeconomic condition* which, once satisfied, ensures full employment of the labour force.

As to the price system, it can be written as:

$$\mathbf{p}^T = w\mathbf{a}_{ni}^T + \mathbf{p}^T\mathbf{A} + \mathbf{p}^T\mathbf{A}\pi \tag{3.11}$$

i.e.:

$$\mathbf{p}^T(\mathbf{I} - \mathbf{A}(1+\pi)) - w\mathbf{a}_{ni}^T = 0 \tag{3.12}$$

We can now follow the same procedure adopted above for the physical quantity system — namely that of characterising a situation of equilibrium — and add a further equation describing a situation of *full expenditure* of total income:

$$wx_n + \mathbf{p}^T\mathbf{A}\pi\mathbf{q} = \mathbf{p}^T\mathbf{y} \tag{3.13}$$

i.e.:

$$-\mathbf{p}^T(\mathbf{I} - \mathbf{A}(1+\pi))(\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}_{in} + w = 0 \tag{3.14}$$

Total wages and total profits must be completely spent. Since we are in a stationary system, in which no new investments are made, the only expenditure recipient is represented by consumption goods.

The price system can thus be stated, in matrix form, as:

$$\begin{bmatrix} \mathbf{p}^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} - \mathbf{A}(1+\pi) & -(\mathbf{I} - \mathbf{A}(1+\pi))(\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}_{in} \\ -\mathbf{a}_{ni}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & 0 \end{bmatrix} \tag{3.15}$$

or as an eigenproblem:[9]

$$\begin{cases} \overline{\mathbf{p}}^T(\lambda_p\overline{\mathbf{I}} - \overline{\mathbf{A}}_p) = \overline{\mathbf{0}} \\ \lambda_p^* = 1 \end{cases} \tag{3.17}$$

[8]This also implies that $(\mathbf{I} - \mathbf{A})^{-1}$ is non-negative, i.e. that its maximum eigenvalue, $\lambda_A^{max}$, satisfies $\lambda_A^{max} < 1$

[9]Where:

$$\overline{\mathbf{A}}_p = \begin{bmatrix} \mathbf{A}(1+\pi) & (\mathbf{I} - \mathbf{A}(1+\pi))(\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}_{in} \\ \mathbf{a}_{ni}^T & 0 \end{bmatrix} \tag{3.16}$$

In this case, matrix $\overline{\mathbf{A}}_p$ has some non-positive elements, i.e. off-diagonal elements of matrix $(\mathbf{I} - \mathbf{A}(1+\pi))$. Therefore, we will proceed stating the conditions for $\lambda_p^* = 1$ to be an eigenvalue of matrix $\overline{\mathbf{A}}_p$. Then we will compute the associated eigenvector, and we will derive the conditions for it to be real and non-negative.

The characteristic equation associated to this eigenproblem is:

$$|\mathbf{A}(1+\pi) - \lambda_p \mathbf{I}| \left(-\lambda_p + \mathbf{a}_{ni}^T(\lambda_p \mathbf{I} - \mathbf{A}(1+\pi))^{-1}(\mathbf{I} - \mathbf{A}(1+\pi))(\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}_{in}\right) = 0$$

i.e.:

$$\mathbf{a}_{ni}^T(\lambda_p \mathbf{I} - \mathbf{A}(1+\pi))^{-1}(\mathbf{I} - \mathbf{A}(1+\pi))(\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}_{in} = \lambda_p \tag{3.18}$$

When $\lambda_p = \lambda_p^*$, expression (3.18) reduces to:

$$\mathbf{a}_{ni}^T(\mathbf{I} - \mathbf{A}(1+\pi))^{-1}(\mathbf{I} - \mathbf{A}(1+\pi))(\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}_{in} = 1 \tag{3.19}$$

i.e.

$$\mathbf{a}_{ni}^T(\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}_{in} = 1 \tag{3.20}$$

which is precisely the same condition as the one previously found for the quantity system. Mathematically, it is again a condition for non-trivial solutions to exist. Economically, it is a *macroeconomic condition* for full expenditure (and, from the quantity system, for full employment of the labour force).

Also in this case, in order for the solution vector to be completely determined, we have to fix arbitrarily one component. Since here no magnitude is exogenously given, as it was the case for total population within the quantity system, determining the scale of the solution means choosing a *numéraire* for the price system. Clearly, such a *numéraire* can be any commodity, or composite commodity, whose price has to be taken as given. In this case, we choose labour as the *numéraire* commodity for the price system, therefore setting $w = \overline{w}$.

The solutions for commodity prices therefore are:

$$\left[\begin{array}{cc} \mathbf{p}^T & w \end{array}\right] = \left[\begin{array}{cc} \overline{w}\mathbf{a}_{ni}^T(\mathbf{I} - \mathbf{A}(1+\pi))^{-1} & \overline{w} \end{array}\right] \tag{3.21}$$

The condition for them to be non-negative is:

$$\pi^{max} \leq \frac{1 - \lambda_A^{max}}{\lambda_A^{max}}$$

where $\lambda_A^{max}$ is the maximum eigenvalue of matrix $\mathbf{A}$.

## 3.2 A growing system

Let us now make the assumption that population grows at the constant, exogenous rate $g \geq 0$, and that per-capita demand for commodity $i$ *as a consumption good* is growing at the rate $r_i \lesseqgtr 0$, $(i = 1, 2, \ldots, m)$. At the aggregate level, therefore, demand for commodity $i$ as a consumption good grows at the rate $(g + r_i)$, $(i = 1, 2, \ldots, m)$.

The total quantities to be produced in period $t$ must now satisfy final demand for consumption goods, replace worn out productive capacity *and* expand it through *new investments*.

In this case, thus, the net output is given by both demand for consumption and demand for new investments:

$$\mathbf{y} = \mathbf{x} + \mathbf{j}$$

where $x_i = a_{in} x_n$ and $j_i = a_{k_i n} x_n$.

The quantity system, in this case, is given by:

$$\begin{bmatrix} \mathbf{I} - \mathbf{A} & -(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ -\mathbf{a}_{ni}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \tag{3.22}$$

and therefore expression (3.8) becomes:

$$\mathbf{a}_{ni}^T (\mathbf{I} - \mathbf{A})^{-1} (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) = 1 \tag{3.23}$$

the solutions being:

$$\begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} = \begin{bmatrix} (\mathbf{I} - \mathbf{A})^{-1} (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{3.24}$$

The price system can be written as:

$$\begin{bmatrix} \mathbf{p}^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} - \mathbf{A}(1+\pi) & -(\mathbf{I} - \mathbf{A}(1+\pi))(\mathbf{I} - \mathbf{A})^{-1} (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ -\mathbf{a}_{ni}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & 0 \end{bmatrix} \tag{3.25}$$

and expression (3.20) becomes:

$$\mathbf{a}_{in}^T (\mathbf{I} - \mathbf{A})^{-1} (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) = 1 \tag{3.26}$$

the solutions being:

$$\begin{bmatrix} \mathbf{p}^T & w \end{bmatrix} = \begin{bmatrix} \overline{w} \mathbf{a}_{ni}^T (\mathbf{I} - \mathbf{A}(1+\pi))^{-1} & \overline{w} \end{bmatrix} \tag{3.27}$$

As it can be seen, while gross quantities are different with respect to the stationary case, having to include new investments too, prices are still the same.

# 4 Vertically integrated and hyper-integrated sectors

When introducing growth in the picture, a crucial role is played by new investments, which are part of the net output in the *current* period, and re-enter the circular flow, as intermediate commodities to be used up by the production process, in the *following* one.

As we are going to see in a moment, the way of treating new investments — and therefore of defining the *net output* — is the key difference between Pasinetti's (1973) and Pasinetti's (1988) approach, i.e. between vertically integrated and vertically *hyper*-integrated analysis.

## 4.1 Vertically integrated sectors — Pasinetti (1973)

Following Pasinetti (1973), let us define the notion of *vertically integrated* sectors.

The net product of the economy is given by

$$\mathbf{y} = \mathbf{x} + \mathbf{j} \tag{4.1}$$

where $\mathbf{x}$'s $i$-th element is the quantity of commodity $i$ demanded as a consumption good, and $\mathbf{j}$'s $i$-th element is the quantity of commodity $i$ demanded as *net* investment. Each vertically integrated sector therefore has, as its final output, a quantity $y_i$ of commodity $i$, sold both for consumption ($x_i$) and for new investment ($j_i$) purposes. Such investment is considered as *exogenous* with respect to technology, and therefore investment goods are treated in the same way as consumption goods.

For each particular $y_i$, we can write:

$$\mathbf{q}^{(i)} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}^{(i)} \tag{4.2}$$

$$\mathbf{s}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}^{(i)} = \mathbf{H}\mathbf{y}^{(i)} \tag{4.3}$$

$$x_n^{(i)} = \mathbf{a}_{ni}^T\mathbf{q}^{(i)} = \mathbf{a}_{ni}^T(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}^{(i)} = \mathbf{v}^T\mathbf{y}^{(i)} \tag{4.4}$$

where $\mathbf{y}^{(i)} = \widehat{\mathbf{y}}\mathbf{e}^{(i)}$.

Matrix $\mathbf{H} = [\mathbf{h_i}]$, in expression (4.3), is the matrix of the units of *vertically integrated productive capacity*, i.e. of *direct and indirect* intermediate requirements for the production of the net product $\mathbf{y}$. The $i$-th column $\mathbf{h}_i$ of such a matrix therefore is a *unit of vertically integrated productive capacity* for vertically integrated sector $i$, i.e. a *composite commodity* made up by *all* the intermediate commodities directly and indirectly required in the whole economic system for the production of one unit of commodity $i$ as net product.

In the same way, row vector $\mathbf{v}^T$, in expression (4.4), is the vector of *vertically integrated labour coefficients*, i.e. the vector of the quantities of labour directly and indirectly employed for the production of one unit of each good entering the net product $\mathbf{y}$.

As $i = 1, \ldots, m$, we have defined $m$ vertically integrated sectors — or *sub-systems*, using Sraffa's terminology — which add up to the complete economic system, and composed by the $i$th element of vector $\mathbf{y}$, the $i$th column of matrix $\mathbf{H}$ and the $i$th element of vector $\mathbf{v}^T$:

> Consider a system of industries (each producing a different commodity) which is in a self-replacing state.
>
> The commodities forming the gross product [...] can be unambiguously distinguished as those which go to replace the means of production and those which together form the net product of the system.
>
> Such a system can be subdivided into as many parts as there are commodities in its net product, in such a way that each part forms a smaller self-replacing system the net product of which consists of only one kind of commodity. These parts we shall call 'sub-systems'.
>
> [...] Although only a fraction of the labour of a sub-system is employed in the industry which directly produces the commodity forming the net product, yet, since all other industries merely provide replacements for the means of production used up, the whole of the labour employed can be regarded as directly or indirectly going to produce that commodity.
>
> (Sraffa 1960, p. 89)

The *gross* quantities produced during the time period by each vertically integrated sector $i$ ($i = 1, 2, \ldots, m$) are given by its net output $y_i = x_i + j_i$ and by a set of intermediate commodities which go to replace those used up during the production process. That part of the net output constituting new investments, $j_i$, will re-enter the circular flow the following period as part of the productive capacity, being distributed to *all* the $m$ vertically integrated sectors according to their — *technologically given* once the rate of growth of demand for consumption goods is known — additional production requirements.

Hence, each vertically integrated sector $i$, in addition to the net product $y_i$, produces the quantities $\mathbf{Aq}^{(i)}$, i.e. the stock of capital goods necessary *at the beginning* of the time period for the production process to take place — and therefore to be *replaced* during the production process itself:

$$\mathbf{s} = \sum_{i=1}^{m} \mathbf{s}^{(i)} = \mathbf{Aq} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} = \mathbf{Hy} \tag{4.5}$$

with

$$\mathbf{s}^{(i)} = \mathbf{Aq}^{(i)} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}^{(i)} = \mathbf{Hy}^{(i)} = \mathbf{h}_i y_i \tag{4.6}$$

where $\mathbf{h}_i$ is the $i$-th column of matrix $\mathbf{H}$.

In the same way, we can express the total amount of labour required for the production of the net output $\mathbf{y}$ as:

$$x_n = \sum_{i=1}^{m} x_n^{(i)} = \mathbf{a}_{ni}^T \mathbf{q} = \mathbf{a}_{ni}^T (\mathbf{I} - \mathbf{A})^{-1} \mathbf{y} = \mathbf{v}^T \mathbf{y} \tag{4.7}$$

with

$$x_n^{(i)} = \mathbf{a}_{ni}^T \mathbf{q}^{(i)} = \mathbf{a}_{ni}^T (\mathbf{I} - \mathbf{A})^{-1} \mathbf{y}^{(i)} = \mathbf{v}^T \mathbf{y}^{(i)} = v_i y_i \tag{4.8}$$

where $v_i$ is the $i$-th element of row vector $\mathbf{v}^T$.

Given these definitions, system (3.2) can be equivalently written as:

$$\mathbf{q} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} + \mathbf{y} = \mathbf{H}\mathbf{y} + \mathbf{y} = (\mathbf{I} + \mathbf{H})\mathbf{y} \tag{4.9}$$

Comparing expressions (3.2) and (4.9), we notice that:[10]

$$(\mathbf{I} - \mathbf{A})^{-1} \equiv (\mathbf{I} + \mathbf{H}) \tag{4.10}$$

Expressions (4.5) and (4.7) can thus be written, respectively, as:

$$\mathbf{s} = \mathbf{A}(\mathbf{I} + \mathbf{H})\mathbf{y} \equiv \mathbf{A}\mathbf{y} + \mathbf{A}\mathbf{H}\mathbf{y} \tag{4.11}$$

i.e. *direct* plus *indirect* capital requirements for the production of net output $\mathbf{y}$, and

$$x_n = \mathbf{a}_{ni}^T (\mathbf{I} + \mathbf{H})\mathbf{y} \equiv \mathbf{a}_{ni}^T \mathbf{y} + \mathbf{a}_{ni}^T \mathbf{H}\mathbf{y} \tag{4.12}$$

i.e. *direct* plus *indirect* labour.

## 4.2 Vertically hyper-integrated sectors — Pasinetti (1988)

In his 1988 paper, Pasinetti adopts a different approach, *generalising* the concept of vertically integrated sectors to that of vertically *hyper*-integrated sectors.

As already hinted at above, the key difference between the two is the way in which *new investment* is treated.

[10] Clearly, this also follows from the series expansion of matrix $(\mathbf{I} - \mathbf{A})^{-1}$:

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots = \mathbf{I} + \mathbf{A}(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots) = \mathbf{I} + \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{H}$$

In Pasinetti (1973), the net product of each vertically integrated sector $i$ is given by $x_i + j_i$, i.e. the quantity of commodity $i$ demanded *both* as a consumption good *and* as a net investment good: new investments are taken as exogenous with respect to technology.

As a consequence, each vertically integrated sector $i$ produces the quantity of commodity $i$ needed by the *whole economic system* as an investment good — and should get from the other sectors the quantities of commodities $j \neq i$ it needs to increase *its own* productive capacity.

On the contrary, Pasinetti (1988) provides a re-definition of the concept of net output, by separating what re-enters the circular flow, namely new investment, from what does not, namely consumption. As a consequence, the net output of a vertically *hyper* integrated sector $i$ is given only by $\mathbf{x}_i$, i.e. the quantity of commodity $i$ demanded as a *consumption* good. New investment is no more considered as exogenous with respect to technology, but as part of it, being determined, in each vertically hyper-integrated sector $i$ ($i = 1, 2, \ldots, m$), by technology itself, once the *growth requirements*, i.e. the rate of growth of final demand for the corresponding consumption commodity, are known. This means that the new investments are determined by the evolution of *both* technological progress *and* final demand.

The gross quantities produced during the time period by each vertically hyper-integrated sector $i$ are therefore given by a quantity $x_i$ of commodity $i$ demanded for consumption purposes, and by a *batch* of intermediate commodities produced both to replace those used up during the production process and to provide the *additional productive capacity* which will be needed at the beginning of the following period in order to satisfy the increased demand for commodity $i$ as a consumption good.

This approach provides us with a *dynamic* generalisation of Sraffa's subsystems: a subsystem sector is defined as "a system of industries [...] which is in a self-replacing state" (Sraffa 1960, p. 89). It should now be clear, however, that a vertically integrated sector is self-replacing only in a single period of time, within a *static* framework. As soon as we introduce growth, the $m$ vertically integrated sectors conforming the economic system as a whole fail to be independent of each other, having to exchange part of their net output — that devoted to new investments — with the others.

On the contrary, vertically hyper-integrated sectors continue to be self-replacing systems through time when growth is introduced, since they produce *all* the intermediate commodities they need not only to replace what has to be used up in the current period to carry on the production process, but also to *expand* their productive capacity in line with the expansion of demand for the corresponding consumption good.

Analytically, the consequences are straightforward. Each vertically hyper-integrated sector grows at its own rate $g + r_i = c_i$ — the rate of change of demand for the consumption good it produces. Following Pasinetti (1988) and Pasinetti (1989), the total quantities to be produced by each 'hyper-subsystem' — or *growing subsystem* — $i$ are given by:

$$\mathbf{q}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} + \mathbf{A}c_i\mathbf{q}^{(i)} + \mathbf{x}^{(i)} \tag{4.13}$$

i.e.:

$$\mathbf{q}^{(i)} = (\mathbf{I} - \mathbf{H}c_i)^{-1}(\mathbf{I} + \mathbf{H})\mathbf{x}^{(i)} \tag{4.14}$$

At the aggregate level, total quantities $\mathbf{q}$ are given by the sum of the sectoral quantities $\mathbf{q}^{(i)}$, i.e:

$$\mathbf{q} = \sum_{i=1}^{m} \mathbf{q}^{(i)} = \sum_{i=1}^{m} (\mathbf{I} - \mathbf{H}c_i)^{-1}(\mathbf{I} + \mathbf{H})\mathbf{x}^{(i)} \tag{4.15}$$

As shown below in appendix A, expression (4.15) can equivalently be written, under certain conditions, as:

$$\mathbf{q} = (\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} \tag{4.16}$$

Using these definitions, we can derive the expressions for sectoral and aggregate capital stocks and labour employment.

The aggregate and sectoral capital stocks are given by:

$$\mathbf{s} = \sum_{i=1}^{m} \mathbf{s}^{(i)} = \mathbf{A}\mathbf{q} = \mathbf{H}(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} = \mathbf{M}\mathbf{x} \tag{4.17}$$

with

$$\mathbf{s}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} = \mathbf{H}(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{M}^{(i)}\mathbf{x}^{(i)} \tag{4.18}$$

or, since $(\mathbf{I} - \mathbf{H}c_i)^{-1} = \mathbf{I} + \mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}$:

$$\begin{aligned} \mathbf{s}^{(i)} &= \mathbf{A}(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \left(\mathbf{A}(\mathbf{I} - \mathbf{H}c_i)^{-1} + \mathbf{A}\mathbf{H}(\mathbf{I} - \mathbf{H}c_i)^{-1}\right)\mathbf{x}^{(i)} = \\ &= \mathbf{A}(\mathbf{I} + c_i\mathbf{M}^{(i)})\mathbf{x}^{(i)} + \mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)} + c_i\mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)} \end{aligned} \tag{4.19}$$

At the beginning of the time period, therefore, each vertically hyper-integrated sector $i$ needs to be provided with a productive capacity which is the sum of three components:

- Intermediate commodities directly required for the production of commodity $i$ as a consumption good — *direct productive capacity* $\mathbf{A}\mathbf{x}^{(i)}$;
- Intermediate commodities directly required for the replacement of those intermediate commodities which will be used up, in the whole vertically hyper-integrated sector, during the production process — *indirect productive capacity* $\mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)}$;
- Intermediate commodities directly required for the expansion of productive capacity according to the over-all increase in the demand for commodity $i$ as a consumption good — *hyper-indirect productive capacity* $c_i\mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)}$.

Thus, $\mathbf{M}$ is the matrix of direct, indirect and hyper-indirect *aggregate* productive capacity for the production of one unit of each commodity entering final demand for consumption goods $\mathbf{x}$. Matrices $\mathbf{M}^{(i)}$ are the matrices of vertically hyper-integrated productive capacity. More specifically, $\mathbf{m}_i^*$, i.e. the $i$-th column of $\mathbf{M}^{(i)}$, is a unit of vertically hyper-integrated productive capacity for the corresponding vertically hyper-integrated sector $i$.

Symmetrically, the aggregate and sectoral quantities of employed labour are given by:

$$x_n = \sum_{i=1}^{m} x_n^{(i)} = \mathbf{a}_{ni}^T\mathbf{q} = \mathbf{v}^T(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} = \mathbf{z}^T\mathbf{x} \tag{4.20}$$

with

$$\begin{aligned} x_n^{(i)} &= \mathbf{a}_{ni}^T\mathbf{q}^{(i)} = \mathbf{v}^T(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{z}^{(i)T}\mathbf{x}^{(i)} \\ &= \mathbf{a}_{ni}^T\mathbf{x}^{(i)} + \mathbf{a}_{ni}^T\mathbf{M}^{(i)}\mathbf{x}^{(i)} + c_i\mathbf{a}_{ni}^T\mathbf{M}^{(i)}\mathbf{x}^{(i)} \end{aligned} \tag{4.21}$$

where $\mathbf{z}^T$ is the vector of aggregate direct, indirect and hyper-indirect labour, and $z_i^*$, i.e. the $i$-th component of each vector $\mathbf{z}^{(i)T}$, is the vertically hyper-integrated labour coefficient for sector $i$.

## 5 Structural change and economic growth

In *Structural Change and Economic Growth* Pasinetti himself states that "all production processes will be considered as vertically integrated" (Pasinetti 1981, p. 29), and that "the notion of 'vertically integrated sectors, which is here used, has been generalised in my article 'Vertical Integration in Economic Analysis', *Metroeconomica*, 1973" (Pasinetti 1981, p. 29n). All sectors are split up into two parts, i.e. a final industry producing the net output — consisting of the *consumption* good — and a 'vertically integrated' industry producing the capital goods directly, indirectly and hyper-indirectly needed by the former.

But now that the difference between vertically integrated and hyper-integrated sectors has been made clear, it should be straightforward to conclude that Pasinetti (1981) framework is actually formulated in vertically *hyper*-integrated terms. In fact, the net output is made up *only* by consumption goods; new investments commodities are produced together with the intermediate ones used up during the production process and therefore to be replaced. Thus, new investments are treated here as in Pasinetti (1988): they are *all* the capital goods — in this case one homogeneous commodity due to the particular simplifying assumptions made on the technique in use — needed by the final industry to expand its productive capacity in order to produce, period after period, the quantity of commodity $i$ demanded as a *consumption good*; their production takes place at the vertically (hyper-)integrated level, i.e. in the capital goods industry, not in the final one, and each subsystem is independent of all the others, producing all intermediate commodities it needs, without buying anything from or selling anything to the others.[11]

In what follows we will first give a synthetic exposition of Pasinetti's (1981) original formulation,[12] and then try to re-state both Pasinetti (1973) and Pasinetti (1988) in the same analytical terms, in order to show that Pasinetti's (1981) approach is a vertically *hyper*-integrated one.

## 5.1 Pasinetti's formulation

In *Structural Change and Economic Growth*, Pasinetti adopts a step-by-step approach: he first presents a pure labour model, in which all production activities are carried out with labour alone — the system produces consumption goods only. Then, he extends the framework by adding capital goods, which are used together with labour for the production of consumption goods, but whose production again requires labour alone — we shall refer to this 'version' of the model as the *intermediate case*. Finally, he presents what he defines the more general version of the

[11] As argued elsewhere (Garbellini & Wirkierman 2010b, section 6), even if a complete and explicit recognition of the notion of vertical hyper-integration has been reached and exposed only in Pasinetti (1988), the idea had already emerged in 1977. The way in which new investments are treated clearly shows that, though not always explicitly stated, Pasinetti's (1981) sectors actually are vertically *hyper*-integrated.

[12] Though with the simplification of considering only stocks of *circulating* capital, in order to avoid further complications and keep the analysis as simple as possible. See footnote 3.

framework, in which both consumption and capital goods are produced by means of both labour and capital goods.

This last version of the model, anyway, has been left aside by Pasinetti (1981) himself — the most important results are developed also for this case, but mainly in footnotes, and the focus is entirely on the intermediate step.

I will follow here exactly the same procedure, by briefly exposing the intermediate version of Pasinetti's (1981) framework.[13] In this case, however, the reason for doing so is a very specific one. As will be shown later on,[14] in all Pasinetti's (1981) formulations, productive capacity is measured in terms of units of *direct* productive capacity, that is to say, the amount of intermediate commodities *directly* required for the production of one unit of a certain commodity. But, due to the particular simplifying assumptions adopted, there is no *analytical* — even if a fundamental and deep *conceptual* — difference, in the intermediate case, between direct, indirect, and hyper-indirect productive capacity, since capital goods are produced by means of labour alone. Therefore, it is particularly convenient to adopt this formulation, since it is straightforward to interpret the units of productive capacity as vertically hyper-integrated ones, and therefore to read the main results in these terms.

It is my contention that this reading key is useful first of all to fully understand how far reaching Pasinetti's (1981) work is. Many implications have not been fully grasped before due to the failure in understanding its vertically hyper-integrated character. In the second place, it provides a link between Pasinetti (1981) and Pasinetti (1988), allowing to use the more complete analytical formulation of the latter to generalise and extend the conclusions of the former.

Pasinetti's (1981) quantity system, in this intermediate case and in matrix terms, is given by:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\mathbf{I} & \mathbf{I} & -\mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{5.1}$$

where:

(i) $\mathbf{x}$ is the vector of physical quantities of final consumption commodities $i = 1, 2, \ldots, m$;

(ii) $\mathbf{x_k}$ is the vector of physical quantities of intermediate (capital) commodities $k_i = k_1, k_2, \ldots, k_m$ (measured in units of productive capacity). Here, the simplifying assumption is made that each intermediate commodity $k_i$ is specific for the production of the corresponding consumption commodity $i$

[13] A very concise exposition of the more complex case is given in appendix A.4.

[14] And also briefly exposed in Garbellini & Wirkierman (2010b).

— and that intermediate commodities themselves are produced by means of labour alone. As I have already said, with respect to Pasinetti's (1981) original formulation, an additional simplifying assumption is made, i.e. that there is circulating capital only;[15]

(iii) $\mathbf{a}_{in}$ is the vector of demand coefficients for final consumption commodities $i = 1, 2, \ldots, m$;

(iv) $\mathbf{a}_{k_i n}$ is the vector of demand coefficients of intermediate commodities $k_i = k_1, k_2, \ldots, k_m$ for new investment, i.e. of per-capita demand for the *units* of (vertically hyper-integrated) productive capacity;

(v) $\mathbf{a}_{ni}^T$ is the vector of (direct) labour requirements for the production of final consumption commodities $i = 1, 2, \ldots, m$;

(vi) $\mathbf{a}_{nk_i}^T$ is the vector of (direct) labour requirements for the production of intermediate commodities $k_i = k_1, k_2, \ldots, k_m$.

It must be further stressed that intermediate commodities are measured by means of a particular unit of measurement, i.e. *units of vertically hyper-integrated productive capacity*: direct, indirect and hyper-indirect requirements for the production of one unit of commodity $i$ as a consumption good.

System (5.1) is made up by three series of equations.

The first one concerns consumption goods, the quantities of which are determined by consumers' effective demand.

The second one concerns capital goods. The quantity to be produced of each capital good $i$ must be enough to replace worn-out productive capacity and provide for the new investment commodities demanded by the final sector.

The last equation is the full-labour-employment one.

The price system is given by:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p_k}^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -(\mathbf{I} + \widehat{\boldsymbol{\pi}}) & \mathbf{I} & \widehat{\boldsymbol{\pi}}\mathbf{a}_{in} - \mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \tag{5.3}$$

where $\mathbf{p}^T$ is the vector of consumption commodities prices, $\mathbf{p_k}^T$ is the vector of intermediate commodities prices and $\widehat{\boldsymbol{\pi}}$ is a diagonal matrix with the sectoral rates of profit on the main diagonal.

[15] Pasinetti, on the contrary, considers fixed capital also. Therefore, his physical quantity coefficient matrix would be:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\widehat{\mathbf{T}}^{-1} & \mathbf{I} & -\mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{5.2}$$

where $T_i^{-1}$ is the depreciation rate for (vertically hyper-integrated) sector $i$.

Both the quantity and the price system are linear and homogeneous systems of equations, and can be written as eigenproblems:[16]

$$\begin{cases} (\mathbf{A_x} - \lambda_x \mathbf{I})\mathbf{x} = \mathbf{0} \\ \lambda_x^* = 1 \\ \lambda_x^* = \lambda_x^{max} \end{cases} \tag{5.6}$$

for the physical quantity system; and:

$$\begin{cases} \mathbf{p}^T(\mathbf{A_p} - \lambda_p \mathbf{I}) = \mathbf{0}^T \\ \lambda_p^* = 1 \\ \lambda_p^* = \lambda_p^{max} \end{cases} \tag{5.7}$$

for the commodity price system.

As to the quantity system, the characteristic equation associated to expression (5.6) is:

$$\begin{aligned} &\begin{vmatrix} -\lambda_x \mathbf{I} & \mathbf{O} \\ \mathbf{I} & -\lambda_x \mathbf{I} \end{vmatrix} \left( -\lambda - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T \end{bmatrix} \begin{bmatrix} -\lambda_x \mathbf{I} & \mathbf{O} \\ \mathbf{I} & -\lambda_x \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} \\ \mathbf{a}_{k_i n} \end{bmatrix} \right) = \\ &= \lambda_x^{2m} \left( -\lambda_x + \frac{1}{\lambda_x} \mathbf{a}_{ni}^T \mathbf{a}_{in} + \frac{1}{\lambda_x^2} \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \frac{1}{\lambda_x} \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} \right) = \\ &= \lambda_x^{2m-2} \left( -\lambda_x^3 + \lambda_x (\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n}) + \mathbf{a}_{nk_i}^T \mathbf{a}_{in} \right) = 0 \end{aligned}$$

Therefore, the first $2m-2$ eigenvalues are repeated eigenvalues equal to zero. The remaining three eigenvalues are the solution of the equation obtained by setting the third degree polynomial in brackets in expression (5.8) equal to zero:

$$(-\lambda_x^3 + \lambda_x (\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n}) + \mathbf{a}_{nk_i}^T \mathbf{a}_{in}) = 0 \tag{5.8}$$

What we are left to do now is to find out the conditions for $\lambda_x^*$ to be one root of this equation, and then to show that, once such condition is satisfied, the two other solutions are smaller than $\lambda_x^*$ itself.

[16] Where

$$\mathbf{A_x} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} \\ \mathbf{I} & \mathbf{O} & \mathbf{a}_{k_i n} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T & 0 \end{bmatrix} \tag{5.4}$$

and:

$$\mathbf{A_p} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} \\ \mathbf{I} + \widehat{\boldsymbol{\pi}} & \mathbf{O} & \mathbf{a}_{k_i n} - \widehat{\boldsymbol{\pi}} \mathbf{a}_{in} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T & 0 \end{bmatrix} \tag{5.5}$$

Finding the condition for $\lambda_x^* = 1$ to be a solution of equation (5.8), means finding the condition that, if satisfied, allows us to find three scalars $a$, $b$ and $c$ such that equation (5.8) can be written as:

$$(\lambda_x - 1)(a\lambda_x^2 + b\lambda_x + c)$$

By using Ruffini's rule, the condition for being able to decompose the third degree polynomial in equation (5.8) in this way emerges as:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} = 1 \tag{5.9}$$

This condition being satisfied, we can finally write:

$$(\lambda_x - 1)\left(-\lambda_x^2 - \lambda_x - \mathbf{a}_{nk_i}^T \mathbf{a}_{in}\right) \tag{5.10}$$

or, equivalently:

$$(\lambda_x - 1)\left(-\lambda_x^2 - \lambda_x - 1 + \mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n}\right) \tag{5.11}$$

The last thing that we have to show is thus that both solutions of the second degree polynomial in brackets are smaller than one. In principle, in computing such solutions, we should consider all the possible cases as to the discriminant of the second degree polynomial, i.e.:

1. $\Delta < 0$: we have two complex solutions;
2. $\Delta > 0$: we have two real, distinct solutions;
3. $\Delta = 0$: we have two real, repeated solutions.

However, we are looking for *real* eigenvalues, and therefore we can rule out case 1 and focus attention on cases 2 and 3.

In the simplest case, i.e. when $\Delta = 0$,[17] in which two repeated eigenvalues are equal to $-1/2 < 1$.

Finally, consider the case in which $\Delta > 0$; here we have two distinct solutions, i.e.:

$$\lambda_q^{1,2} = \frac{-1 \pm \sqrt{\Delta}}{2} = \frac{-1 \pm \sqrt{1 - 4\mathbf{a}_{nk_i}^T \mathbf{a}_{in}}}{2}$$

Clearly, we are interested in the greater one only, which is not greater than 1 when:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} \leq 3 \tag{5.12}$$

[17] For this to be true, the demand and direct labour coefficients must be such that $\mathbf{a}_{nk_i}^T \mathbf{a}_{in} = 0.25$. Clearly, this is a very special case.

or, equivalently, when:

$$\mathbf{a}_{nk_i}^T \mathbf{a}_{in} \geq -2 \tag{5.13}$$

i.e. in all economically meaningful cases.

Hence, when condition (5.9) is satisfied, $\lambda_x^* = 1 = \lambda_x^M$. Such a condition is the *macroeconomic condition* for full employment of the labour force, and it is the sum of three addenda:

- $\mathbf{a}_{ni}^T \mathbf{a}_{in}$: direct labour required for the production of consumption commodities — *direct labour*;
- $\mathbf{a}_{nk_i}^T \mathbf{a}_{in}$: direct labour required for replacing the units of productive capacity used up during the production process — *indirect labour*;
- $\mathbf{a}_{nk_i}^T \mathbf{a}_{k_in}$: direct labour required for the production of the units of productive capacity demanded as new investment commodities, i.e. in order to expand productive capacity — *hyper-indirect labour*.

The vector of physical quantities for consumption and intermediate commodities, therefore, is the right-hand-side eigenvector associated to $\lambda_x^* = 1$, which is completely determined once we fix one component — in this case, following Pasinetti (1981), once we set $x_n = \overline{x}_n$:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{in}\overline{x}_n \\ \mathbf{x} + \mathbf{a}_{k_in}\overline{x}_n \\ \overline{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{in}\overline{x}_n \\ (\mathbf{a}_{in} + \mathbf{a}_{k_in})\overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{5.14}$$

As to the price system, we have first of all to notice that, in order for matrix $\mathbf{A_p}$ to be non-negative, the following condition should hold:

$$\widehat{\boldsymbol{\pi}} \leq (\widehat{\mathbf{a}_{k_in}^{-1}})\mathbf{a}_{in}$$

or:

$$\pi_i \leq \frac{a_{in}}{a_{k_in}}, \qquad i = 1, 2, \ldots, m \tag{5.15}$$

which means that the number of units of productive capacity that, evaluated at current prices, give us the profit component of prices is smaller than or at most equal to the number of units of final consumption commodities to be produced during the production process. Since the profit component of prices defines the amount of value created in excess with respect to replacements (and wages), this condition being satisfied would imply that the realised profits could allow, *at most*, to produce *in each sector* a number of units of productive capacity exactly equal to that required by the expansion of productive capacity in line with the evolution of final demand for consumption commodities.[18]

This is not necessarily so. Therefore, we will follow here the same procedure followed in section 3 to solve the industry-level price system, i.e. that of looking for the condition(s) guaranteeing that $\lambda_p^* = 1$ be an eigenvalue of matrix $\overline{\mathbf{A}}_p$, then computing the associated left-hand eigenvector, and then again finding out the conditions for this vector to be non-negative.

The characteristic equation associated to expression (5.7) is:

$$\begin{vmatrix} -\lambda_p \mathbf{I} & \mathbf{O} \\ \mathbf{I} + \widehat{\boldsymbol{\pi}} & -\lambda_p \mathbf{I} \end{vmatrix} \left( -\lambda_p - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T \end{bmatrix} \begin{bmatrix} -\lambda_p \mathbf{I} & \mathbf{O} \\ \mathbf{I} + \widehat{\boldsymbol{\pi}} & -\lambda_p \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} \\ \mathbf{a}_{k_i n} \end{bmatrix} \right) =$$
$$= \lambda_p^{2m} \left( -\lambda_p + \frac{1}{\lambda_p} \mathbf{a}_{ni}^T \mathbf{a}_{in} + \frac{1}{\lambda_p^2} \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \frac{1}{\lambda_p^2} \mathbf{a}_{nk_i}^T \mathbf{a}_{in} - \frac{1}{\lambda_p} \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \frac{1}{\lambda_p} \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} \right) =$$
$$= \lambda_p^{2m-2} \left( -\lambda_p^3 + \lambda_p \mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \widehat{\boldsymbol{\pi}} \mathbf{a}_{in} - \lambda_p \mathbf{a}_{nk_i}^T \widehat{\boldsymbol{\pi}} \mathbf{a}_{in} + \lambda_p \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} \right) = 0 \tag{5.16}$$

What we want to do now is finding the condition that makes $\lambda^* = 1$ an eigenvalue of matrix $\overline{\mathbf{A}}_p$, i.e. to be a solution of equation (5.16). For this to be true, by substituting $\lambda_p = \lambda_p^*$ into (5.16), the expression in brackets must be equal to zero.

Therefore, the condition for $\lambda_p^* = 1$ to be an eigenvalue of matrix $\overline{\mathbf{A}}_p$ is:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} = 1 \tag{5.17}$$

which is exactly the same condition as (5.9), obtained above for $\lambda_x^* = 1$ to be an eigenvalue of matrix $\overline{\mathbf{A}}_x$.

Hence, expression (5.9) is a necessary condition for the price system also to have non-trivial solutions, and moreover, it is a *macroeconomic condition* for full expenditure of income and — as we know from the quantity system — for full employment of the labour force.

The vector of commodity prices is the left-hand-side eigenvector of matrix $\mathbf{A}_p$ associated to $\lambda_p^* = 1$, and is completely determined once one component is arbitrarily fixed. In this case, this amounts to choosing a *numéraire* for the price system;

[18] As will become clear later on, after having provided the single-period equilibrium conditions, and assuming that all sectoral profits are invested, this requirement is satisfied in stock-equilibrium. It is therefore also satisfied in a condition of stock disequilibrium characterised by a lack of productive capacity. It would not be satisfied though in a situation of stock disequilibrium with non-utilised productive capacity, which is anyway a perfectly possible situation.

again following Pasinetti (1981), we chose labour as the *numéraire* commodity, therefore setting $w = \overline{w}$, and obtaining:

$$\begin{bmatrix} \mathbf{p}^T \\ \mathbf{p_k}^T \\ w \end{bmatrix}^T = \begin{bmatrix} \overline{w}\mathbf{a}_{ni}^T + \mathbf{p_k}^T(\mathbf{I} + \widehat{\boldsymbol{\pi}}) \\ \overline{w}\mathbf{a}_{nk_i}^T \\ \overline{w} \end{bmatrix}^T = \begin{bmatrix} \overline{w}(\mathbf{a}_{ni}^T + \mathbf{a}_{nk_i}^T(\mathbf{I} + \widehat{\boldsymbol{\pi}})) \\ \overline{w}\mathbf{a}_{nk_i}^T \\ \overline{w} \end{bmatrix}^T \tag{5.18}$$

which is always non-negative provided that

$$\pi_i \geq -\frac{a_{ni} + a_{nk_i}}{a_{nk_i}}, \quad \forall i = 1, 2, \ldots, m$$

Now that we have the solution vectors for physical quantities and commodity prices, we can analyse in more details the (single-period) *equilibrium conditions*.

We have already said what the word 'equilibrium' means within Pasinetti's (1981) framework: it is a situation in which labour force is fully employed, income is fully spent and *productive capacity is fully utilised*. Macroeconomic condition (5.9) concerns the *flows* of the economic system, and guarantees to comply with the first two equilibrium requirements.

The third equilibrium requirement, on the contrary, concerns the *stocks* of the economic system: each vertically hyper-integrated sector must be provided, at the beginning of the time period, with the number of units of productive capacity allowing it to carry on the production process in line with final demand requirements. Hence, we do not have a single condition, but rather a *series* of *sectoral conditions*. Before stating them, we must accordingly introduce a new series of *sectoral* magnitudes:

$$\mathbf{k} = [k_i], \qquad i = 1, 2, \ldots, m$$

where $k_i$ is the number of units of (vertically hyper-integrated) productive capacity necessary *at the beginning of the production process* for it to be carried out.

Therefore, in order for productive capacity to be fully utilised, the following series of sectoral conditions must be satisfied:

$$\mathbf{k} = \mathbf{x} \tag{5.19}$$

The statement of macroeconomic condition (5.9) and of sectoral conditions (5.19) closes the exposition of Pasinetti's (1981) framework analysing production in the short run.

For the purposes of the present paper, this is all we need to know about Pasinetti's (1981) analytical formulation in order to compare it with Pasinetti (1973) and Pasinetti (1988).

## 5.2 Vertically integrated sectors

We now want to re-state Pasinetti's (1973) framework in terms formally analogous to Pasinetti's (1981) formulation. In doing so, we have to take into account the already mentioned major differences between the two as to the description of the technique.

In Pasinetti (1981), a specific commodity is *either* a consumption good *or* an intermediate commodity; moreover, each specific capital good $k_i$ is only devoted to the production of the corresponding consumption commodity $i$. The only *inter-industry* flows are therefore those going from industry $k_i$ to industry $i$ $(i = 1, 2, \ldots, m)$, and each sector is conformed by two industries.

In Pasinetti (1973) — and also in Pasinetti (1988) — on the contrary, any of the $m$ commodities produced in the economic system can be used *both* as a consumption good *and* as an intermediate commodity. Therefore, there is no neat distinction, in general, between consumption and intermediate commodities. Such a distinction arises only within each vertically integrated sector $i$, where only commodity $i$ is produced as net output, while *all* commodities (included commodity $i$ itself) are produced as intermediate commodities. In a few words, each commodity $i$ appears as a final commodity only in the corresponding sector $i$, while it appears as an intermediate commodity in *all* sectors.

It is still possible to think of particular intermediate commodities specific to each sector, but of course in this case they will be *composite* commodities, whose constituent elements are the same in all sectors, though entering them in *sector-specific proportions*.

Moreover, as already mentioned above, in Pasinetti (1973) the net product of each sector is made up by the sum of two components: $x_i$, i.e. the quantity of commodity $i$ demanded as a *consumption good*, and $j_i$, i.e. the quantity of commodity $i$ demanded as an *investment commodity*. In this way, vertically integrated sector $i$ produces a part of *its own* new productive capacity, i.e. the $i$-th component, *and* a part of that of *all other sectors*. Therefore, the batch of commodities to be devoted to new investment are not produced together with the capital goods, but together with, and 'in the same way as', the consumption goods.

In terms of inter-industry and inter-sectoral flows, all inter-industry relations are reintroduced, and there are also some inter-sectoral flows, all sectors selling to the others part of their net product, and buying from all the others part of their net product, in order to build up new productive capacity.

Hence, the physical quantity system is:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ -\mathbf{I} & \mathbf{I} & \mathbf{O} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{ni}^T \mathbf{H} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x}_k \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{5.20}$$

where $\mathbf{a}_{ni}^T\mathbf{H} = \mathbf{a}_{nk_i}^T$ and $a_{k_in}$ is per-capita demand for commodity $i$ as an investment good.

The system stated above can also be written as an eigenproblem as follows:[19]

$$\begin{cases} (\lambda_x \overline{\mathbf{I}} - \overline{\mathbf{A}}_x)\overline{\mathbf{q}} = \overline{\mathbf{0}} \\ \lambda_x^* = 1 \\ \lambda_x^* = \lambda_x^{max} \end{cases} \tag{5.22}$$

the characteristic equation being:

$$\left| \begin{array}{cc} -\lambda_x \mathbf{I} & \mathbf{O} \\ \mathbf{I} & -\lambda_x \mathbf{I} \end{array} \right| \left( -\lambda_x - \left[ \begin{array}{cc} \mathbf{a}_{ni}^T & \mathbf{a}_{ni}^T\mathbf{H} \end{array} \right] \left[ \begin{array}{cc} -\lambda_x \mathbf{I} & \mathbf{O} \\ \mathbf{I} & -\lambda_x \mathbf{I} \end{array} \right]^{-1} \left[ \begin{array}{c} \mathbf{a}_{in} + \mathbf{a}_{k_in} \\ \mathbf{0} \end{array} \right] \right) = 0 \tag{5.23}$$

Solving and rearranging we get:

$$(\lambda_x)^{2m} \left( -\lambda_x + \frac{1}{\lambda_x}\mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in}) + \frac{1}{\lambda_x^2}\mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_in}) \right) = 0 \tag{5.24}$$

or:

$$(\lambda_x)^{2m-2} \left( -\lambda_x^3 + \lambda_x \mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in}) + \mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_in}) \right) = 0 \tag{5.25}$$

The characteristic equation associated to this eigenproblem has $2m-2$ repeated roots equal to zero. We are left with three other possibly *real* eigenvalues, the solutions to the polynomial in the second brackets.

If

$$\mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in}) + \mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_in}) = 1$$

i.e.:

$$\mathbf{a}_{ni}^T(\mathbf{I} + \mathbf{H})(\mathbf{a}_{in} + \mathbf{a}_{k_in}) \equiv \mathbf{v}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in}) = 1 \tag{5.26}$$

then expression (5.25) can be re-written as:

$$\lambda_x^{2m-1}(\lambda_x^* - 1)(\lambda_x^2 + \lambda_x + 1 - \mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in})) = 0 \tag{5.27}$$

The solution resulting from the first expression in brackets is precisely $\lambda_x^* = 1$, while the last two are the solutions of the second degree equation in the second brackets. If real, i.e. if:

$$\mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_in}) < \frac{3}{4}$$

[19] Where:

$$\overline{\mathbf{A}}_x = \left[ \begin{array}{ccc} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} + \mathbf{a}_{k_in} \\ \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{ni}^T\mathbf{H} & 0 \end{array} \right] \tag{5.21}$$

we want them not to be greater than one; this would imply:

$$\mathbf{a}_{ni}^{T}\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \geq -2$$

which of course is true in all economically meaningful cases.

Expression (5.26) is the *macroeconomic condition* for full-employment of the labour force, analogous to expression (5.9) found out in the previous section for Pasinetti's (1981) original framework. In this case, however, we can see that it is the sum of *two* components:

- $\mathbf{a}_{ni}^{T}(\mathbf{a}_{in} + \mathbf{a}_{k_i n})$: labour directly needed for the production of consumption *and* new investment commodities: *direct labour*;
- $\mathbf{a}_{ni}^{T}\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_i n})$: labour directly needed for the replacement of the intermediate commodities used up during the production process: *indirect labour*.

Once we set $x_n = \overline{x}_n$, the right-hand eigenvector associated to $\lambda_x^* = \lambda_x^{max} = 1$ gives us the solutions for physical quantities, i.e.:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{x}_k \\ x_n \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_{in} + \mathbf{a}_{k_i n})\overline{x}_n \\ (\mathbf{a}_{in} + \mathbf{a}_{k_i n})\overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{5.28}$$

As to the price system, it can be written as:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} & -(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ -\pi\mathbf{I} & \mathbf{I} - \mathbf{H}\pi & \pi(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ -\mathbf{v}^T & -\mathbf{v}^T\mathbf{H} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \tag{5.29}$$

or, in eigen form, as:[20]

$$\begin{cases} \overline{\mathbf{p}}^T(\lambda_p^*\overline{\mathbf{I}} - \overline{\mathbf{A}}_p) = \overline{\mathbf{0}}^T \\ \lambda_p^* = 1 \end{cases} \tag{5.31}$$

[20]Where:

$$\overline{\mathbf{A}}_p = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} + \mathbf{a}_{k_i n} \\ \pi\mathbf{I} & \mathbf{H}\pi & -\pi(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ \mathbf{v}^T & \mathbf{v}^T\mathbf{H} & 0 \end{bmatrix} \tag{5.30}$$

Here we do not want $\lambda_p^* = 1$ to be the maximum eigenvalue, since the matrix has at least some negative element, and therefore Perron Frobenius theorems do not apply. We will instead follow the same procedure already followed above, namely that of stating the condition for it to be an eigenvalue, in order to compute the associated eigenvector and therefore finding out the conditions for it to be real and non-negative.

The characteristic equation is:

$$\begin{vmatrix} -\lambda_p \mathbf{I} & \mathbf{O} \\ \pi \mathbf{I} & \mathbf{H}\pi - \lambda_p \mathbf{I} \end{vmatrix} \times$$
$$\times \left( -\lambda_p - \begin{bmatrix} \mathbf{v}^T & \mathbf{v}^T \mathbf{H} \end{bmatrix} \begin{bmatrix} -\lambda_p \mathbf{I} & \mathbf{O} \\ \pi \mathbf{I} & \mathbf{H}\pi - \lambda_p \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} + \mathbf{a}_{k_i n} \\ -\pi(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \end{bmatrix} \right) = 0 \tag{5.32}$$

Since the determinant appearing as the first factor of expression (5.32) is:

$$(-1)^m \lambda_p^m d_{\mathbf{H}\pi}$$

where $d_{\mathbf{H}\pi}$ is the determinant of matrix $(\mathbf{H}\pi - \lambda_p^* \mathbf{I})$, and the inverse of such matrix can be written as:

$$\frac{1}{d_{\mathbf{H}\pi}} (\mathbf{H}\pi - \lambda_p \mathbf{I})^{(+)}$$

where $(\mathbf{H}\pi - \lambda_p^* \mathbf{I})^{(+)}$ is the adjoint matrix, the inverse matrix in expression (5.32) can be written as:

$$\begin{bmatrix} -\lambda_p^{m-1} d_{\mathbf{H}\pi} \mathbf{I} & \mathbf{O} \\ \pi \lambda_p^{m-1} (\mathbf{H}\pi - \lambda_p^* \mathbf{I})^{(+)} & \lambda_p^m (\mathbf{H}\pi - \lambda_p \mathbf{I})^{(+)} \end{bmatrix} \tag{5.33}$$

Hence, expression (5.32) becomes, after some manipulations:

$$(-1)^m \Big( -\lambda_p^{m+1} d_{\mathbf{H}\pi} - \lambda_p^{m-1} d_{\mathbf{H}\pi} \mathbf{v}^T (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) - \pi \lambda_p^{m-1} \mathbf{v}^T \mathbf{H} (\mathbf{H}\pi - \lambda_p^* \mathbf{I})^{(+)} (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) +$$
$$+ \pi \lambda_p^m \mathbf{v}^T \mathbf{H} (\mathbf{H}\pi - \lambda_p^* \mathbf{I})^{(+)} (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) = 0 \Big) \tag{5.34}$$

With $\lambda_p^* = 1$ it reduces to:

$$(-1)^m d_{\mathbf{H}\pi} \left( -1 + \mathbf{v}^T (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \right) = 0 \tag{5.35}$$

since up to this point $d_{\mathbf{H}\pi}$, with $\lambda_p^* = 1$, is a scalar which does not depend any more on $\lambda_p$ itself.

Therefore, the condition for $\lambda_p^* = 1$ to be an eigenvalue of matrix $\mathbf{A_p}$ is:

$$\mathbf{v}^T (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) = 1 \tag{5.36}$$

which is precisely the same condition as the one found above for the quantity system, guaranteeing full expenditure of income as well as full employment of the labour force.

What is left to do is to check whether the associated eigenvector is real and non-negative. Such eigenvector, once we set $w = \overline{w}$, is:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} = \begin{bmatrix} \overline{w}\mathbf{v}^T(\mathbf{I} - \mathbf{H}\pi)^{-1} \\ \overline{w}\mathbf{v}^T\mathbf{H}(\mathbf{I} - \mathbf{H}\pi)^{-1} \\ \overline{w} \end{bmatrix}^T \tag{5.37}$$

Using Perron-Frobenius theorems (see also Pasinetti 1977, p. 89) we can conclude that it is non-negative when:

$$\pi^{max} < \frac{1}{\lambda_H^{max}} \tag{5.38}$$

i.e. when (uniform) rate of profit is smaller than the maximum eigenvalue of matrix $\mathbf{H}$.

## 5.3 Vertically hyper-integrated sectors

Before going to the reformulation of Pasinetti's (1988) framework, it is worth spending a few words on the particular unit of measurement used for intermediate commodities. In Pasinetti (1981), *direct* productive capacity is used as a unit of measurement; however, not only such a choice cannot be made here, but it is also possible to read Pasinetti's (1981) framework in vertically hyper-integrated terms.

Going back to section 4.2, the physical quantity system for the $i$-th vertically integrated sector can be written as:

$$\mathbf{q}^{(i)} = (\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \tag{5.39}$$

or

$$\mathbf{q}^{(i)} = \mathbf{x}^{(i)} + \mathbf{H}\mathbf{x}^{(i)} + (\mathbf{I} + \mathbf{H})\mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{x}^{(i)} + \mathbf{q_k}^{(i)} \tag{5.40}$$

where $\mathbf{q_k}^{(i)}$ is the batch of commodities entering vertically hyper-integrated sector $i$'s *gross* investment.

As we have already said, in Pasinetti (1981) the capital goods in each vertically integrated sector are measured in units of direct productive capacity for the corresponding final (i.e. consumption) commodity. This is possible thanks to the assumption according to which such productive capacity consists of a homogeneous capital good. In the more general formulation, this not the case anymore, since productive capacities are *composite* commodities. This entails no *conceptual* difficulty, since the productive capacity of each vertically integrated sector can be seen as a particular composite commodity, in which intermediate goods enter in particular proportions.

The analytical problem arises because the proportions in which the various commodities enter direct, indirect, and hyper-indirect productive capacity are different, so that it is not possible to say that the batch of commodities necessary, for example, for the production of one unit of direct productive capacity for a certain final commodity is a scalar multiple of such productive capacity itself. This could happen only in the very particular case in which the eigenvectors of matrix $\mathbf{A}$ were its own columns.

More specifically, the demand for capital goods in 'traditional' units, using a formulation analogous to Pasinetti's (1981) one, is:

$$\mathbf{q_k}^{(i)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{A}\mathbf{q_k}^{(i)} + \mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \tag{5.41}$$

i.e. as the sum of: the intermediate commodities directly necessary for the production of the final consumption commodities ($\mathbf{A}\mathbf{x}^{(i)}$); the intermediate commodities directly necessary for the production of the whole set of capital goods ($\mathbf{A}\mathbf{q_k}^{(i)}$); and new investment ($\mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}$).[21]

In order to express it in units of *direct* productive capacity for consumption good $i$, such capacity being the $i$-th column of matrix $\mathbf{A}$, we should be able to write the above expression as:

$$\beta_k \mathbf{a}_i x_i = \beta_x \mathbf{a}_i x_i + \beta_{Ak} \mathbf{a}_i x_i + \beta_{Mc_i} \mathbf{a}_i x_n \tag{5.42}$$

with $\beta_k$, $\beta_x$, $\beta_{Ak}$, and $\beta_{Mc_i}$ being all scalars. Direct comparison of the left-hand side of equations (5.41) and (5.42) reveals that this is not possible. Since

$$\mathbf{q_k}^{(i)} = \mathbf{H}(1 + c_i)(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = (\mathbf{I} + \mathbf{H})(1 + c_i)(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{a}_i x_i \tag{5.43}$$

such comparison would imply that:

$$\mathbf{H}(1 + c_i)(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = (1 + c_i)(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{a}_i x_i = \beta_k \mathbf{a}_i x_i \tag{5.44}$$

which could be possible only if $\mathbf{a}_i$ were an eigenvector of matrix $(\mathbf{I}+\mathbf{H})(\mathbf{I}-\mathbf{H}c_i)^{-1}$ and $\beta_k$ the associated eigenvalue. But this matrix, by definition, has exactly the same eigenvectors as matrix $\mathbf{A}$. Therefore, unless in very special cases (e.g. if $\mathbf{A}$ is diagonal, i.e. the very special case considered by Pasinetti (1981)), equivalence (5.44) shall not hold, and direct productive capacity cannot be used as a unit of measurement for capital goods in the vertically hyper-integrated sector.

What we must actually do, therefore, is choosing another unit of measurement for intermediate commodities. While in the previous section, dealing with vertical integration, we solved the problem by using a unit of vertically integrated

[21] See appendix A.2 for details.

productive capacity as the unit of measurement, here the most obvious choice is represented by the units of vertically *hyper*-integrated productive capacity for consumption good $i$.

As to Pasinetti's (1981) formulation, if we stick to the *intermediate case* — i.e. with consumption commodities produced by means of labour and intermediate commodities, and capital goods produced by means of labour alone — we can actually read all the analytical formulations as if they were written in vertically hyper-integrated terms. The reason is very simple: since no capital goods are required for the production of capital goods themselves, there is no difference, in physical terms, between direct, vertically integrated and vertically hyper-integrated productive capacity. In fact, both indirect and hyper-indirect requirements are those intermediate commodities required for the production of capital goods directly used up in the production of the final consumption commodity, either in the current or in future periods, and therefore, in this simplified case, they are simply equal to zero. All productive capacity needed by this simplified economic system therefore reduces to the direct one.

Going back to the general case, the expression for $\mathbf{q_k}^{(i)}$ can be equivalently written as:[22]

$$\mathbf{q_k}^{(i)} = \mathbf{M}^{(i)}\mathbf{x}^{(i)} + c_i\mathbf{M}^{(i)}\mathbf{x}^{(i)} = \mathbf{m}_i^* x_i + c_i \mathbf{m}_i^* a_{in} x_n \tag{5.45}$$

where $\mathbf{m}^*$ is the $i$-th — i.e. the *relevant* — column of matrix $\mathbf{M}^{(i)}$. Therefore, using $\mathbf{m}_i^*$ as the measurement unit, we can write the above expression as:

$$x_{k_i} = x_i + c_i a_{in} x_n \tag{5.46}$$

This means that the number of units of vertically hyper-integrated productive capacity to be produced during the production process ($x_{k_i}$) in sector $i$ ($i = 1, 2, \ldots, m$) is given by the number of units of final consumption commodity $i$ necessary to satisfy final demand ($x_i$) plus the number of units of such a consumption commodity which will be additionally demanded in the following period ($c_i a_{in} x_n$), for which additional productive capacity must be set up.

In order to complete our reformulation of Pasinetti's (1988) quantity system, what is left is the last equation. In particular, we need to specify the meaning of the coefficients $a_{nk_i}$ in the present context. In Pasinetti (1981), such coefficients were the direct labour necessary for the production of one unit of *direct* productive capacity for the consumption good $i$. Since the definition of productive capacity adopted here is that of vertically hyper-integrated productive capacity, these coefficients come to have a different meaning — namely, that of direct labour necessary

[22]See appendix A.2 for details.

for the production of one unit of vertically hyper-integrated productive capacity — and we can accordingly write:

$$a_{nk_i} = \mathbf{a}_{ni}^T \mathbf{m}^{(i)} \tag{5.47}$$

Now we have all the elements we need to write down the physical quantity system which, in matrix form, is:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\mathbf{I} & \mathbf{I} & -\widehat{\mathbf{c}}\mathbf{a}_{in} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{ni}^T \overline{\mathbf{M}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{5.48}$$

where $\overline{\mathbf{M}}$ is a matrix made up by the relevant columns of matrices $\mathbf{M}^{(i)}$, $\forall i = 1, 2, \ldots, m$, i.e. a matrix whose $i$-th column is $\mathbf{m}^{(i)}$.[23]

As usual, we can state expression (5.48) as an eigenproblem:

$$\begin{cases} (\lambda_x \mathbf{I} - \overline{\mathbf{A}}_x)\overline{\mathbf{x}} = \overline{\mathbf{0}} \\ \lambda_x^* = 1 \\ \lambda_x^* = \lambda_x^{max} \end{cases} \tag{5.49}$$

In order for $\overline{\mathbf{x}}$ to be a real and positive eigenvector of the *non-negative* matrix $\overline{\mathbf{A}}_x$, $\lambda_x^* = 1$ must be a solution of the characteristic equation associated to this eigenproblem; more specifically, the *maximum* solution.

The characteristic equation of this eigenproblem is:

$$\lambda_x^{2m} \left( -\lambda_x - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{ni}^T \overline{\mathbf{M}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\lambda_x}\mathbf{I} & \mathbf{O} \\ -\frac{1}{\lambda_x^2}\mathbf{I} & -\frac{1}{\lambda_x}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{in} \\ \widehat{\mathbf{c}}\mathbf{a}_{in} \end{bmatrix} \right) = 0 \tag{5.50}$$

i.e.:

$$\lambda^{2m-2} \left( -\lambda^3 + \lambda(\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}} \widehat{\mathbf{c}} \mathbf{a}_{in}) + \mathbf{a}_{ni}^T \overline{\mathbf{M}} \mathbf{a}_{in} \right) = 0 \tag{5.51}$$

Thus, the first $2m-2$ solutions are all zeros. We are left with the second factor, which is a third degree equation in $\lambda_x$. If

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}} \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}} \widehat{\mathbf{c}} \mathbf{a}_{in} = 1 \tag{5.52}$$

[23] As it is shown in appendix A.1, under certain conditions — the same as the ones making (4.16) equivalent to (4.15) — $\overline{\mathbf{M}}$ is *approximately* equal to $\mathbf{M}$. However, I have preferred to develop all the following elaborations in terms of $\overline{\mathbf{M}}$ rather than $\mathbf{M}$. In view of the possibility of empirical applications of this framework, the fact that they are not precisely the same means that the less disaggregated the data available are, the bigger the difference between the two matrices is. The same arguments will hold for the two vectors $\overline{\mathbf{z}}^T$ and $\mathbf{z}^T$.

such equation can be decomposed in the following way:

$$(\lambda_x^* - 1)(-\lambda_x^2 - \lambda_x - 1 + \mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in}) = 0 \tag{5.53}$$

i.e. we have one solution equal to 1, which is the one we are looking for, and then two other solutions, resulting from the second degree equation in (5.53).

These solutions are real when:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in} \geq \frac{3}{4}, \quad \text{or} \quad \mathbf{a}_{ni}^T \overline{\mathbf{M}}\mathbf{a}_{in} \leq \frac{1}{4}$$

and they are smaller than (or equal to) unity when:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in} \leq 3, \quad \text{or} \quad \mathbf{a}_{ni}^T \overline{\mathbf{M}}\mathbf{a}_{in} \geq -2$$

i.e. in all economically relevant cases.

Expression (5.52) is the *macroeconomic condition* for full employment of the labour force, analogous to expressions (5.9) — for Pasinetti (1981) case — and (5.26) — for Pasinetti (1973) case. Anyway, if there was an asymmetry between (5.9) and (5.26), we can see that such asymmetry has disappeared with respect to (5.52), since we again have the sum of *three* components:

- $\mathbf{a}_{ni}^T \mathbf{a}_{in}$: direct labour for the production of consumption commodities — *direct labour*;
- $\mathbf{a}_{ni}^T \overline{\mathbf{M}}\mathbf{a}_{in}$: direct labour for the replacement of the units of productive capacity used up during the production process — *indirect labour*;
- $\mathbf{a}_{ni}^T \overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in}$: direct labour for the production of the units of productive capacity needed to expand productive capacity in line with the evolution of demand for consumption commodities — *hyper-indirect labour*.

Thus, once condition (5.52) is satisfied, and once we set $x_n = \overline{x}_n$, the right-hand eigenvector associated to $\lambda_x^* = 1$ is the solution vector for physical quantities, i.e.:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_k \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{in}\overline{x}_n \\ (\mathbf{I} + \widehat{\mathbf{c}})\mathbf{a}_{in}\overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{5.54}$$

As to the price system, by following the procedure suggested by Pasinetti (1988, section 4), we may notice that we have $m$ *equivalent* ways of expressing the price system:

$$\begin{gathered} \mathbf{p}^T = w\mathbf{a}_{ni}^T + \mathbf{p}^T\mathbf{A} + \mathbf{p}^T\mathbf{A}\pi \equiv w\mathbf{a}_{ni}^T + \mathbf{p}^T\mathbf{A}(1 + c_i) + \mathbf{p}^T\mathbf{A}(\pi - c_i) \\ \forall i = 1, 2, \ldots, m \end{gathered} \tag{5.55}$$

and hence:

$$\mathbf{p}^T = w\mathbf{z}^{(i)T} + \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i), \qquad \forall i = 1, 2, \ldots, m \tag{5.56}$$

Since the $i$-th element of vector $\mathbf{p}^T$, i.e. the price of commodity $i$, can be written as:

$$p_i = \overline{w}z_i^* + \mathbf{p}^T\mathbf{m}_i^*(\pi - c_i) \tag{5.57}$$

and thus expression (5.56) can be equivalently written also as:

$$\mathbf{p}^T \equiv \overline{w}\overline{\mathbf{z}}^T + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}}) \equiv \overline{w}\mathbf{z}^{(i)T} + \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i) \tag{5.58}$$

Now, vertically hyper-integrated sector $i$ does not produce only commodity $i$, but also all the intermediate commodities that utilises as inputs for producing commodity $i$ itself as a consumption good. This means that the sector also produces the corresponding units of vertically hyper-integrated productive capacity, i.e. $\mathbf{m}_i^*$, *and* all the commodities directly, indirectly and hyper-indirectly necessary for producing these units of productive capacity, and so on.

Therefore, in principle we should derive, for each vertically hyper-integrated sector $i$ ($i = 1, 2, \ldots$), not a single price for composite commodity $\mathbf{m}_i^*$, i.e. $p_{k_i}$, but rather a whole vector of prices $\mathbf{p}_k^{(i)T}$, given by

$$\mathbf{p}_k^{(i)T} = \mathbf{p}^T\mathbf{M}^{(i)}, \qquad i = 1, 2, \ldots, m \tag{5.59}$$

The $i$-th element of such vector is the price of one unit of productive capacity for consumption commodity $i$. But then, all the elements, included the $i$-th one, can be used to compute the price of a unit of productive capacity for productive capacity itself, and for productive capacity for it, and so on.

Consider one unit of vertically hyper-integrated productive capacity for sector $i$:

$$\mathbf{m}_i^* = [m_{ji}^{(i)}], \qquad i = 1, 2, \ldots, m$$

Which is the composite commodity we need to produce *this* composite commodity? For each element $m_{ji}^{(i)}$, we need a quantity $m_{bj}^{(i)}m_{ji}^{(i)}$ of each commodity $b$, with $b = 1, 2, \ldots, m$, used as an intermediate commodity. We thus have a vector of commodities for each element of column vector $\mathbf{m}_i^*$, and hence a matrix; more precisely, matrix $(\mathbf{M}^{(i)})^2$, which must be evaluated at current prices, that is to say, by using expression (5.59):

$$\mathbf{p}^T(\mathbf{M}^{(i)})^2 = \mathbf{p}_k^{(i)}\mathbf{M}^{(i)}$$

The price of one unit of vertically hyper-integrated productive capacity for consumption commodity $i$ therefore is the $i$-th component of vector $\mathbf{p}_k^{(i)T}$:

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}^{(i)T}\mathbf{M}^{(i)} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{5.60}$$

which, defining $\mathbf{z}_k^{(i)T} \equiv \mathbf{z}^{(i)T}\mathbf{M}^{(i)}$ becomes:

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}_k^{(i)T} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{5.61}$$

The price of a unit of vertically hyper-integrated for sector $i$ is the $i$-th element of this vector, i.e.:

$$p_{k_i} = \overline{w}z_{k_i}^* + \mathbf{p}_k^{(i)T}\mathbf{m}_i^*(\pi - c_i) \tag{5.62}$$

where $z_{k_i}^*$ is the $i$-th element of vector $\mathbf{z}_k^{(i)T}$.

Since, however, expression (5.59) holds for all $i = 1, 2, \ldots, m$, we can also write:

$$p_{k_i} = \mathbf{p}^T\mathbf{m}_i^*, \qquad i = 1, 2, \ldots, m \tag{5.63}$$

and therefore

$$\mathbf{p}_k^T \equiv \mathbf{p}^T\overline{\mathbf{M}} \tag{5.64}$$

In this way, equation (5.58) becomes:

$$\mathbf{p}^T = \overline{w}\mathbf{z}^T + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}}) \tag{5.65}$$

Moreover, by substituting expression (5.65) into (5.64), the latter can be written as:

$$\mathbf{p}_k^T = \overline{w}\mathbf{z}^T\overline{\mathbf{M}} + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} = \overline{w}\overline{\mathbf{z}}_k^T + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} \tag{5.66}$$

which means that the $i$-th element of vector $\mathbf{p}_k^T$, i.e. the price of one unit of vertically hyper-integrated productive capacity for consumption commodity $i$, can be written as:

$$p_{k_i} = \overline{w}\overline{\mathbf{z}}^T\mathbf{m}_i^* + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* = \overline{w}z_{k_i} + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* \tag{5.67}$$

where $z_{k_i}$ is the $i$-th element of vector $\overline{\mathbf{z}}_k^T$. Expressions (5.67) and (5.62) are therefore equivalent.[24]

[24]For a comparison between these two expressions, see section 6 below.

By using the expressions for consumption commodity and productive capacity prices, i.e. expressions (5.65) and (5.66), we can now formulate the price system, in matrix form and analogously to Pasinetti's (1981) one, as:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -(\pi\mathbf{I} - \widehat{\mathbf{c}}) & \mathbf{I} - (\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} & (\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{a}_{in} \\ -\overline{\mathbf{z}}^T & -\overline{\mathbf{z}}^T\overline{\mathbf{M}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \tag{5.68}$$

or, as an eigenproblem, as:[25]

$$\begin{cases} \overline{\mathbf{p}}^T(\lambda_p\overline{\mathbf{I}} - \overline{\mathbf{A}}_p) = \overline{\mathbf{0}}^T \\ \lambda_p^* = 1 \end{cases} \tag{5.70}$$

the characteristic equation being:

$$\begin{vmatrix} -\lambda_p\mathbf{I} & \mathbf{O} \\ \pi\mathbf{I} - \widehat{\boldsymbol{c}} & (\pi\mathbf{I} - \widehat{\boldsymbol{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I} \end{vmatrix} \times$$
$$\times \left( -\lambda_p - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{ni}^T\overline{\mathbf{M}} \end{bmatrix} \begin{bmatrix} -\lambda_p\mathbf{I} & \mathbf{O} \\ \pi\mathbf{I} - \widehat{\boldsymbol{c}} & (\pi\mathbf{I} - \widehat{\boldsymbol{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} \\ (\widehat{\mathbf{c}} - \pi\mathbf{I})\mathbf{a}_{in} \end{bmatrix} \right) = 0 \tag{5.71}$$

By defining $d_{\overline{\mathbf{M}}\pi} = |(\pi\mathbf{I} - \widehat{\boldsymbol{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I}|$, the first factor reduces to $(-1)^m\lambda_p^m d_{\overline{\mathbf{M}}\pi}$ and the inverse matrix in the second factor can be written as:

$$\begin{bmatrix} -\frac{1}{\lambda_p} & \mathbf{O} \\ -\frac{1}{\lambda_p}\frac{1}{d_{\overline{\mathbf{M}}\pi}}((\pi\mathbf{I} - \widehat{\boldsymbol{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)}(\pi\mathbf{I} - \widehat{\mathbf{c}}) & \frac{1}{d_{\overline{\mathbf{M}}\pi}}((\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)} \end{bmatrix}$$

Hence, by substituting these last two results into the characteristic equation, the latter can be written as:

$$(-1)^m \Big( -\lambda^{m+1}d_{\overline{\mathbf{M}}\pi} + \lambda_p^{m-1}\mathbf{z}^T\mathbf{a}_{in} + \lambda^{m-1}\mathbf{z}^T\overline{\mathbf{M}}((\pi\mathbf{I} - \widehat{\boldsymbol{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{a}_{in} +$$
$$-\lambda_p^m\mathbf{z}^T\overline{\mathbf{M}}((\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{a}_{in} \Big) = 0 \tag{5.72}$$

[25] Where:

$$\overline{\mathbf{A}}_p^T = \begin{bmatrix} \mathbf{O} & \overline{\mathbf{O}} & \mathbf{a}_{in} \\ \pi\mathbf{I} - \widehat{\mathbf{c}} & (\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} & -(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{a}_{in} \\ \overline{\mathbf{z}}^T & \overline{\mathbf{z}}^T\overline{\mathbf{M}} & 0 \end{bmatrix} \tag{5.69}$$

Since the matrix has at least one negative element, we cannot use the Perron Frobenius theorems. We will therefore use the same procedure as in the previous section.

When $\lambda_p^* = 1$ the above expression reduces to:

$$(-1)^m \Big(-\overline{d}_{\overline{\mathbf{M}}\pi} + \overline{d}_{\overline{\mathbf{M}}\pi}\mathbf{z}^T\mathbf{a}_{in} + \mathbf{z}^T\overline{\mathbf{M}}((\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)}(\pi\mathbf{I}-\widehat{\mathbf{c}})\mathbf{a}_{in} + \\ -\mathbf{z}^T\overline{\mathbf{M}}((\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)}(\pi\mathbf{I}-\widehat{\mathbf{c}})\mathbf{a}_{in}\Big) = 0 \tag{5.73}$$

i.e:

$$(-1)^m\overline{d}_{\overline{\mathbf{M}}\pi}\left(-1 + \mathbf{z}^T\mathbf{a}_{in}\right) = 0 \tag{5.74}$$

and therefore:

$$\overline{\mathbf{z}}^T\mathbf{a}_{in} = 1 \tag{5.75}$$

i.e. the same condition derived from the eigenproblem associated to the quantity system, now also guaranteeing full expenditure of income.

The left-hand eigenvector associated to this eigenvalue is the solution for commodity prices, i.e., with $w = \overline{w}$:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} = \begin{bmatrix} \overline{w}\left[\overline{\mathbf{z}}^T\left(\mathbf{I} - \overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}})\right)^{-1}\right] \\ \overline{w}\left[\overline{\mathbf{z}}^T\overline{\mathbf{M}}\left(\mathbf{I} - (\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}}\right)^{-1}\right] \\ \overline{w} \end{bmatrix}^T = \begin{bmatrix} \overline{w}\overline{\mathbf{z}}^T\mathbf{\Phi}(\pi) \\ \overline{w}\overline{\mathbf{z}}_k^T\mathbf{\Phi}_k(\pi) \\ \overline{w} \end{bmatrix}^T \tag{5.76}$$

where $\overline{\mathbf{z}}_k^T = \overline{\mathbf{z}}^T\overline{\mathbf{M}}$, $\mathbf{\Phi}(\pi) = \left(\mathbf{I} - \overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}})\right)^{-1}$ and $\mathbf{\Phi}_k(\pi) = \left(\mathbf{I} - (\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}}\right)^{-1}$.

Notice that since $\overline{\mathbf{p}}_k^T$ can be written either as $\mathbf{p}^T\overline{\mathbf{M}}$ or as $\overline{\mathbf{z}}_k^T\mathbf{\Phi}_k(\pi)$, it follows that

$$\overline{\mathbf{z}}^T\mathbf{\Phi}(\pi)\overline{\mathbf{M}} = \overline{\mathbf{z}}^T\overline{\mathbf{M}}\mathbf{\Phi}_k(\pi) \tag{5.77}$$

and therefore that

$$\mathbf{\Phi}(\pi) = \overline{\mathbf{M}}\mathbf{\Phi}_k(\pi)\overline{\mathbf{M}}^{-1} \tag{5.78}$$

i.e. $\mathbf{\Phi}(\pi)$ and $\mathbf{\Phi}_k(\pi)$ *similar matrices*. Thus they have the same eigenvalues, and their eigenvectors, call them $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_k$, respectively, are such that $\boldsymbol{\theta} = \overline{\mathbf{M}}\boldsymbol{\theta}_k\overline{\mathbf{M}}^{-1}$.

Since we want these solutions to be non-negative, we first have to check the condition guaranteeing non-negativity of matrix $\overline{\mathbf{M}}$.[26]

[26] As we can see from appendix A.1, such condition is:

$$c_i \leq \frac{1}{\lambda_H^{max}}, \qquad \forall i = 1, 2, \ldots, m \tag{5.79}$$

Then, in order for $(\pi\mathbf{I} - \widehat{\mathbf{c}})$ to be non-negative, all $c_i$'s must be smaller than, or equal to, the rate of profit. This ensures that $\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})$ is non-negative. We can now use the Perron Frobenius theorems to conclude that for prices to be non-negative the maximum eigenvalue of matrix $\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})$ must be smaller than or equal to 1.

The results for all cases considered so far are summarised in table 1.

## 6 The price system

It is worth devoting some time to the analysis of prices (5.76) in comparison to those derived by Pasinetti (1981, pp. 41-3) for the intermediate case — or, exploiting the result obtained in Garbellini & Wirkierman (2010b, p. 15), in comparison to:

$$p_i = \overline{w}\left(\ell_i + a_{nk_i}(\pi_i - (g + r_i))\right) \tag{6.1}$$

As Pasinetti points out:

> [O]ur approach has made it possible to express *all* price components in such a way as to allow the wage rate to be factored out. This means that what appears in the square brackets, by being multiplied by the wage rate, must obviously be either a physical quantity of labour or something which is made to be equivalent to a physical quantity of labour. [...] [L]et us notice that — whatever the way in which the rates of profit are determined, the [(5.76)] imply a theory of value which is based on quantities of physical labour and on quantities which are made to be equivalent to physical labour. The prices thereby express a theory of value which is indeed no longer in terms of pure labour, but in terms of what we may call *labour equivalents*.
>
> (Pasinetti 1981, pp. 42-3)

Pasinetti thus characterises the theory of value coming from formulation (5.76) as a *labour-equivalents theory of value*, as opposed to the pure labour one. With Pasinetti's (1981) simplified description of the technique in use, this is reflected by the fact that indirect (and hyper-indirect) labour is weighted more than direct one when a positive rate of profit is present.

Once inter-industry relations are considered, the adoption of 'labour-equivalents' values is slightly more complicated. The pure labour value of commodity $i$ ($i = 1, 2, \ldots, m$) would be given by $\overline{w}z_i^*$ — i.e. by the wage rate multiplied by the quantity of vertically hyper-integrated labour necessary for the production of one unit of commodity $i$ as a consumption good — while its labour equivalent value is given by the wage rate multiplied by a *linear combination* of *all* the labour coefficients,

| | Physical quantities | Commodity prices | Macroeconomic condition | Non-negativity condition (prices) |
|---|---|---|---|---|
| Industry-level | $\mathbf{q}=(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n$ <br> $x_n=\overline{x}_n$ | $\mathbf{p}^T=\overline{w}\mathbf{a}_{ni}^T(\mathbf{I}-\mathbf{A}(1+\pi))^{-1}$ <br> $w=\overline{w}$ | $\mathbf{a}^T(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}_{in}+\mathbf{a}_{k_in})=1$ | $\pi^{max}\leq\frac{1-\lambda_A^{max}}{\lambda_A^{max}}$ |
| Pasinetti (1981) | $\mathbf{x}=\mathbf{a}_{in}\overline{x}_n$ <br> $\mathbf{x}_k=(\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n$ <br> $x_n=\overline{x}_n$ | $\mathbf{p}^T=\overline{w}(\mathbf{a}_{ni}^T+\mathbf{a}_{nk_i}^T(\mathbf{I}+\widehat{\boldsymbol{\pi}}))$ <br> $\mathbf{p}_k^T=\overline{w}\mathbf{a}_{nk_i}^T$ <br> $w=\overline{w}$ | $\mathbf{a}_{ni}^T\mathbf{a}_{in}+\mathbf{a}_{nk_i}^T\mathbf{a}_{in}+\mathbf{a}_{nk_i}^T\mathbf{a}_{k_in}=1$ | $\pi_i\geq-\frac{a_{ni}+a_{nk_i}}{a_{nk_i}}$ |
| Pasinetti (1973) | $\mathbf{y}=(\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n$ <br> $\mathbf{x}_k=(\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n$ <br> $x_n=\overline{x}_n$ | $\mathbf{p}^T=\overline{w}\mathbf{v}^T(\mathbf{I}-\mathbf{H}\pi)^{-1}$ <br> $\mathbf{p}^T=\overline{w}\mathbf{v}^T\mathbf{H}(\mathbf{I}-\mathbf{H}\pi)^{-1}$ <br> $w=\overline{w}$ | $\mathbf{a}_{ni}^T(\mathbf{a}_{in}+\mathbf{a}_{k_in})+\mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in}+\mathbf{a}_{k_in})=1$ <br> or <br> $\mathbf{v}^T(\mathbf{a}_{in}+\mathbf{a}_{k_in})=1$ | $\pi^{max}=\frac{1}{\lambda_H^{max}}$ |
| Pasinetti (1988) | $\mathbf{x}=\mathbf{a}_{in}\overline{x}_n$ <br> $\mathbf{x}_k=(\mathbf{I}+\widehat{\mathbf{c}})\mathbf{a}_{in}\overline{x}_n$ <br> $x_n=\overline{x}_n$ | $w=\overline{w}\mathbf{z}^T(\mathbf{I}-\overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}}))^{-1}$ <br> $w=\overline{w}\mathbf{z}^T\overline{\mathbf{M}}(\mathbf{I}-(\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}})^{-1}$ <br> $w=\overline{w}$ | $\mathbf{a}_{ni}^T\mathbf{a}_{in}+\mathbf{a}_{ni}^T\overline{\mathbf{M}}\mathbf{a}_{in}+\mathbf{a}_{ni}^T\overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in}=1$ <br> or <br> $\mathbf{z}^T\mathbf{a}_{in}=1$ | $c_i^{max}\leq\frac{1}{\lambda_H^{max}}$ |

Table 1: Price and quantity systems, their solutions, macroeconomic conditions, and non-negativity conditions for commodity prices, in the four cases considered.

the multipliers being the elements of the corresponding column of matrix $\mathbf{\Phi}(\pi)$ — call it $\boldsymbol{\phi}_i(\pi)$, $(i = 1, 2, \ldots, m)$.

When considering the price of the units of vertically hyper-integrated productive capacity, things become more complicated. As stated in section 5.3 (page 72) above, there are two equivalent ways to express such prices, given by expressions (5.67) and (5.62):

$$p_{k_i} = \overline{w}\mathbf{z}^{(i)T}\mathbf{m}_i^* + \mathbf{p}_k^{(i)T}\mathbf{m}_i^*(\pi - c_i) = \overline{w}z_{k_i}^* + \mathbf{p}_k^{(i)T}\mathbf{m}_i^*(\pi - c_i) \tag{5.62}$$

$$p_{k_i} = \overline{w}\overline{\mathbf{z}}^T\mathbf{m}_i^* + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* = \overline{w}z_{k_i} + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* \tag{5.67}$$

These two expressions are equivalent, but of course the two addenda constituting them are not the same.

In expression (5.62), both the (vertically hyper-integrated) labour cost and the profit components are exactly those associated to the production of composite commodity $[m_{1i}, m_{2i}, \ldots, m_{mi}]$ by vertically hyper-integrated sector $i$ as a unit of productive capacity. On the contrary, in expression (5.67) both of them are those that we *would* have, were each of these quantities produced in the vertically hyper-integrated sector producing the corresponding commodity as a *final consumption* commodity. This is a major difference. The hyper-indirect part of each vertically hyper-integrated sector $i$ crucially depends on the rate of growth of the sector itself, i.e. on the rate of growth of demand for the corresponding consumption good. Therefore, expression (5.67) does under- or over-estimate labour costs according to whether[27]

$$c_i \lesseqgtr \frac{\mathbf{v}^T\mathbf{D}^{(i)}\widehat{\mathbf{c}}\mathbf{m}_i^*}{\mathbf{v}^T\mathbf{D}^{(i)}\mathbf{m}_i^*}, \qquad i = 1, 2, \ldots, m \tag{6.2}$$

therefore over- or under-estimating correspondingly the profit component of price $p_{k_i}$.

Therefore, the labour value of a unit of vertically hyper-integrated productive capacity for sector $i$ is not given by $\overline{w}\overline{\mathbf{z}}^T\overline{\mathbf{M}}$, but by $\overline{w}\mathbf{z}(i)_T\mathbf{M}^{(i)}$; in the solutions obtained in section 5.3, i.e. the second expression in (5.76), therefore, matrix $\mathbf{\Phi}_k(\pi)$ does not transform the true *labour values* into prices of production.

The correct formulation for the whole set of intermediate commodities prices, as hinted at in the previous section, includes the whole price *vector* for each vertically hyper-integrated sector; specifically, for each vertically hyper-integrated sector $i$, it is given by equation (5.60):

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}^{(i)T}\mathbf{M}^{(i)} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{5.60}$$

[27]Matrix $\mathbf{D}$ is the first derivative of matrix $(\mathbf{I} - \mathbf{H}c_i)^{-1}$ with respect to $c_i$. For details, see appendix A.3.

i.e.

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}_k^{(i)T} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{6.3}$$

and therefore

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}_k^{(i)T}(\mathbf{I} - \mathbf{M}(\pi - c_i))^{-1} = \overline{w}\mathbf{z}_k^{(i)T}\mathbf{\Phi}^{(i)}(\pi) \tag{6.4}$$

Matrix $\mathbf{\Phi}^{(i)}(\pi)$ is the matrix that, *in each vertically hyper-integrated sector* $i$, transforms the labour values for intermediate commodities into the respective prices of production. The $i$-th element of each vector $\mathbf{p}_k^{(i)T}$ clearly is equivalent to the corresponding element of vector $\mathbf{p}_k^T$.

We can therefore call matrices $\mathbf{\Phi}(\pi)$ and $\mathbf{\Phi}^{(i)}(\pi)$ the *labour transformation* matrices, the scalar $z_i^e = \overline{\mathbf{z}}^T\boldsymbol{\phi}_i(\pi)$ the *labour equivalent* for the production of consumption commodity $i$, and the scalar $z_{k_i}^e(\pi) = \overline{\mathbf{z}}^T\overline{\mathbf{M}}\boldsymbol{\phi}_{k_i}(\pi)$ — or equivalently $z_{k_i}^e(\pi) = \mathbf{z}^{(i)}\mathbf{M}^{(i)}\boldsymbol{\phi}_{k_i}^{(i)}(\pi)$ — the *labour equivalent* for the production of one unit of productive capacity for vertically hyper-integrated sector $i$. With this new notation, prices can be now written as:

$$\begin{cases} p_i = \overline{w}z_i^e(\pi) \\ p_{k_i} = \overline{w}z_{k_i}^e(\pi) \end{cases} \quad , \quad i = 1, 2, \ldots, m \tag{6.5}$$

When the rate of change of demand is different from sector to sector, and with a uniform, exogenously given, rate of profit, prices and (labour) values are therefore diverging, due to the difference between the future rate of growth of sectoral demands for consumption goods and the rate of profit. This difference originates shifts of value among the various sectors, which aim at allowing each of them to keep satisfied its physical (i.e. quantity side) requirements for equilibrium growth *given* the distributive variables.

## 7 Sectoral and aggregate magnitudes through time

The last step of this discussion about production in the short run consists in explicitly stating the analytical formulation of three magnitudes, both in sectoral and aggregate terms, which are bound to acquire great importance in Pasinetti's (1981) treatment of economic dynamics: capital/output ratio(s), capital/labour ratio(s), and product per worker.

As Pasinetti (1981) explains in depth (for details, see Pasinetti 1981, chapter IX, sections 4-6),[28] it is very important to stress the conceptual difference between

[28] And as will become clear when dealing with production in the long run, i.e. with the general dynamic model (Garbellini 2010a).

the capital/output ratio and the capital/labour ratio. In a few words, they both are an index of the 'roundaboutness' of a production process, but while the first ratio expresses the degree of *capital intensity* — and therefore is relevant, among the other things, for the process of *price formation* — the second one expresses the corresponding *degree of mechanisation*, and therefore is relevant for problems concerning *employment*.

Let us start from the capital/output ratio. The $m$ *sectoral* ratios are given by:

$$\gamma_i = \frac{p_{k_i} k_i}{p_i x_i} = \frac{p_{k_i} x_i}{p_i x_i} = \frac{\overline{w}\mathbf{z}_k^T \boldsymbol{\phi}_{k_i}(\pi)}{\overline{w}\mathbf{z}^T \boldsymbol{\phi}_i(\pi)} \equiv \frac{\overline{w}\mathbf{z}_k^{(i)T} \boldsymbol{\phi}_i^{(i)}(\pi)}{\overline{w}\mathbf{z}^T \boldsymbol{\phi}_i(\pi)} \equiv \frac{z_{k_i}^e(\pi)}{z_i^e(\pi)} \tag{7.1}$$

where the second equality comes from the fact that here we are assuming to be in stock-equilibrium, i.e. that $k_i = x_i$ for all $i = 1, 2, \ldots, m$.[29]

Looking at expression (7.1), we can see that the *sectoral* capital/output ratios are the ratios of two quantities of labour equivalents. The wage rate appears both in the numerator and in the denominator, and therefore cancels out. Thus, such ratios only depend on technology, and on the rate of profit — or better, on the difference between it and the sectoral rates of growth:

> [...] the incidence of capital in each commodity price, i.e. that component of each price which has to be charged for the use of capital, is proportional to the capital/output ratio required in that production process, quite independently of the number or the value of machines operated by each worker. The lower the capital/output ratio, the lower the charge for capital in each price, no matter whether and how much the capital/labour ratio may be changing.
>
> (Pasinetti 1981, p. 181)

This can be seen more clearly by writing the price of consumption commodity $i$ ($i = 1, 2, \ldots, m$) as:

$$p_i = \overline{w} z_i^* + \frac{p_{k_i}}{p_i} p_i (\pi - c_i)$$

and therefore:

$$p_i = \overline{w} z_i^* + \gamma_i p_i (\pi - c_i) \tag{7.2}$$

The second addendum in expression (7.2) is precisely the charge for capital in the price of consumption commodity $i$, directly proportional to the corresponding capital/output ratio.

[29] See Garbellini & Wirkierman (2010b, section 3.3) for details.

The *aggregate* capital/output ratio, on the other hand, is given by:

$$\Gamma = \frac{\mathbf{p}_k^T\mathbf{k}}{\mathbf{p}^T\mathbf{x}} = \frac{\mathbf{p}_k^T\mathbf{x}}{\mathbf{p}^T\mathbf{x}} = \frac{\overline{w}\overline{\mathbf{z}}_k^T\mathbf{\Phi}_k(\pi)\mathbf{a}_{in}}{\overline{w}\overline{\mathbf{z}}^T\mathbf{\Phi}(\pi)\mathbf{a}_{in}} = \frac{\mathbf{z}_{k_i}^e(\pi)^T\mathbf{a}_{in}}{\mathbf{z}_i^e(\pi)^T\mathbf{a}_{in}} \tag{7.3}$$

As it appears clearly by looking at expression (7.3), the *aggregate* degree of capital intensity depends not only on technology and on the rate of profit, but also on the *structure* of final demand for consumption commodities. Changing the *composition* of final demand, therefore, makes the capital/output ratio of the economic system as a whole change, even if technology and income distribution are still the same.

The meaning of the capital/labour ratio is entirely a different one. The $m$ *sectoral* ratios are given by:

$$\theta_i = \frac{p_{k_i}k_i}{z_i^* x_i} = \frac{p_{k_i}x_i}{z_i^* x_i} = \frac{\overline{w}\overline{\mathbf{z}}_k^T\boldsymbol{\phi}_{k_i}(\pi)}{z_i^*} \equiv \frac{\overline{w}\mathbf{z}_k^{(i)T}\boldsymbol{\phi}_i^{(i)}(\pi)}{z_i^*} \equiv \frac{\overline{w}z_{k_i}^e(\pi)}{z_i^*} \tag{7.4}$$

The capital/labour ratio for vertically hyper-integrated sector $i$ ($i = 1, 2, \ldots, m$) is the ratio between a *stock* of capital, evaluated at current prices, and a *flow* of labour. In this case, as it is apparent from expression (7.4), the wage rate only appears in the numerator, so that it does not cancel out. Hence, the sectoral degree of mechanisation depends on technology, on the rate of profit, *and* on the wage rate.

The *aggregate* capital/labour ratio, on the other hand, is given by:

$$\Theta = \frac{\mathbf{p}_k^T\mathbf{k}}{\mathbf{z}^T\mathbf{x}} = \frac{\mathbf{p}_k^T\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} = \frac{\overline{w}\overline{\mathbf{z}}_k^T\mathbf{\Phi}_k(\pi)\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} \equiv \frac{\overline{w}\mathbf{z}_k^{e^T}(\pi)\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} \tag{7.5}$$

Also in this case, as for the capital/output ratio, the aggregate expression (7.5) also depends on the *composition* of final demand for consumption goods.

We can now have a look at another quite important economic magnitude, i.e. the product per worker, sectoral and aggregate, respectively:

$$y_i = \frac{p_t x_i}{z_i^* x_i} = \frac{\overline{w}\overline{\mathbf{z}}^T\boldsymbol{\phi}_i(\pi)}{z_i^*} \equiv \frac{\overline{w}z_i^e(\pi)}{z_i^*} \tag{7.6}$$

and:

$$Y_t = \frac{\mathbf{p}^T\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} = \frac{\overline{w}\overline{\mathbf{z}}^T\mathbf{\Phi}(\pi)\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} \equiv \frac{\overline{w}\mathbf{z}^{e^T}(\pi)\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} \tag{7.7}$$

Also in this case, the difference between expression (7.6) and expression (7.7) lies in the fact that the *sectoral* product per worker does not depend on the structure of final demand for consumption commodities, while the *aggregate* one does.

## 8 Conclusions

As stated in the Introduction, the aim of the present paper was first of all that of stressing the vertically hyper-integrated character of Pasinetti's (1981) framework, by underlining the conceptual and analytical analogies with Pasinetti (1988) as to the treatment of new investment and, therefore, to the definition of net output.

Secondly, I have tried to reformulate the first part of Pasinetti's (1981) book, that concerning production in the short run, by removing some simplifying assumptions on the technology in use, by using matrix algebra, and by restating both the quantity and the price system as eigenproblems. This algebraic reformulation is intended to be a first step towards a complete generalisation of the whole analysis carried out by Pasinetti (1981) on the one side — by taking full advantage of Pasinetti (1988) generalisation — and to make it possible to apply this framework for empirical and institutional analyses on the other side.

In particular, I would like to devote some lines to the discussion of three aspects which I regard as particularly relevant.

First, the deepest implications of this approach can be fully drawn when the more realistic characterisation of the technique in use is re-introduced. The complex network of inter-industry relations is in fact an aspect of primary importance of modern economic systems; disregarding it prevents us from grasping the main potentialities of this approach as to its ability of making us understand, explain, and eventually look for a way to change, reality. Thanks to vertical hyper-integration, this aspect can be re-introduced into the picture without losing the possibility of keeping the analysis itself at the most fundamental level. This task is accomplished by using vertically hyper-integrated productive capacities as the units of measurement for intermediate commodities.

In this way, capital accumulation can be studied with respect to the final consumption commodities produced in the $m$ sectors conforming the economic system, leaving the problem of their *composition* aside, but keeping the possibility of resuming it in any moment — vertically hyper-integrated analysis simply entails a *linear* algebraic transformation of usual inter-industry matrices, which can be reverted without any problem.[30]

Second, both the physical quantity and the commodity price systems can be restated as eigenproblems, the solutions being the eigenvectors associated to a specific eigenvalue. The macroeconomic condition emerges as the condition for such a value to be actually an eigenvalue of the coefficient matrix, and therefore to get economically meaningful solutions out of these eigensystems. The restatement fol-

[30]The implications of using vertically hyper-integration for the study of capital accumulation will be explored in Garbellini (2010a), where I reformule that part of Pasinetti's (1981) book dealing with production in the long run.

lows from the reformulation, using matrix algebra — and in particular partitioned matrices — of the two equation systems. It is a more compact, and therefore easier to manage, mathematical formulation with respect to that adopted by Pasinetti (1981) and Pasinetti (1988).

Matrix algebra is a powerful mathematical tool, and it should be possible to take advantage of its further utilisation within Modern Classical economic theory. It is a matter of fact that some problems which seemed unsolvable to many Classical economists — think of Ricardo and Marx — actually were so only because of the lack of proper mathematical tools. Hence, in general, trying to restate 'old' problems in 'new' ways is one of the main keys to be able to successfully re-switch back to Classical Political Economy.

Moreover, restating the quantity and price systems as eigensystems makes it easier to work out empirical applications, since there are many numerical techniques, to be performed with the main statistical software, for the computation of such measures: solving eigenproblems is a very straightforward way of managing actual data. In addition eigenvalues — though not eigenvectors — possess the property of surviving similarity transformations of matrices, which means that conclusions can be reached from their analysis even if we have inter-industry matrices in nominal, rather than in physical, terms.[31]

Third, I have tried to do a step forward with respect to the analytical formulation in Pasinetti (1988). In the latter, Pasinetti defines matrices $\mathbf{M}^{(i)}$, $(i = 1, 2, \ldots, m)$, i.e. the matrices of vertically hyper-integrated productive capacity for each sector $i$. But, he says, each of such matrices has only one relevant column, i.e. the $i$-th one. Such a column vector is that composite commodity that he calls a unit of vertically hyper-integrated productive capacity for sector $i$.

Then, Pasinetti (1988) uses such matrices for expressing the price system in vertically hyper-integrated terms, i.e. reformulating it in order to make the role of vertically hyper-integrated productive capacity explicit. However, this is done in order to reach the definition of natural rates of profit; we have $m$ price systems, one for each vertically hyper-integrated sector $i$, each of them with its own natural rate of profit $\pi_i^*$, in each of them appearing the corresponding matrix $\mathbf{M}^{(i)}$, whose columns, in this particular case, are *all*, not only the $i$-th one, relevant.

The notion of natural rates of profit has not yet been introduced here. Since it is a notion closely related to capital accumulation, and therefore to dynamics, I have defined and explored it in the paper devoted to this topic, i.e. Garbellini (2010a). In the present paper I have stated, in section 5.3, the price system with

[31] The lack of proper, i.e. physical, data when performing empirical analyses is an old problem. Clearly, such a restatement does not solve it. But using matrix algebra and exploring its possible further applications can be a step forward in the right direction.

a (uniform[32]) *exogenous* rate of profit, keeping its determination as a degree of freedom. I have therefore introduced a more general formulation of the price system, defining a matrix $\overline{\mathbf{M}}$ made up by the $i$-th column of each of matrix $\mathbf{M}^{(i)}$ ($i = 1, 2, \ldots, m$) and using it for the formulation of the price system in terms analogous to what Pasinetti does in *Structural Change and Economic Growth*.

By doing so, it emerges clearly that, for the determination of (vertically hyper-integrated) sectoral intermediate commodity prices, *all* columns of each matrix $\mathbf{M}^{(i)}$ are relevant, not only within the 'natural' economic system, but also in general. In the general case, however, there is an equivalence between the 'outcomes' of the price system formulated in terms of matrix $\overline{\mathbf{M}}$ and in terms of matrices $\mathbf{M}^{(i)}$. We can therefore rely on the former, using the latter when dealing with issues requiring particular sectoral considerations. As we will see in Garbellini (2010a), this is not true anymore when non-uniform *sectoral* rates of profit — e.g. the 'natural' ones — are introduced.

Further explanations and algebraic proofs are given in the Appendix below.

# A Appendix

## A.1 Growing subsystems and aggregate quantities

Following Pasinetti (1989), we can write the quantity system, with non-proportional growth, as:

$$\mathbf{q}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} + \mathbf{A}(g + r_i)\mathbf{q}^{(i)} + \mathbf{x}^{(i)} \tag{A.1}$$

i.e.:

$$\mathbf{q}^{(i)} = \mathbf{H}c_i\mathbf{q}^{(i)} + (\mathbf{I} + \mathbf{H})\mathbf{x}^{(i)} \tag{A.2}$$

and therefore:

$$\mathbf{q}^{(i)} = (\mathbf{I} - \mathbf{H}c_i)^{-1}(\mathbf{I} + \mathbf{H})\mathbf{x}^{(i)} \tag{A.3}$$

Quantities (A.3) are non-negative provided that matrix $(\mathbf{I} - \mathbf{H}c_i)^{-1}$ is non-negative, i.e., for Perron-Frobenius theorems, that the maximum eigenvalue of matrix $\mathbf{H}$, $\lambda_H^{max}$, is such that

$$c_i < \frac{1}{\lambda_H^{max}}, \qquad \forall i = 1, 2, \ldots, m \tag{A.4}$$

[32] Pasinetti (1981) directly formulates the price system using non uniform ones. However, this would have, in the present framework, complicated the algebraic formulation without adding anything to the main conclusions, since the introduction of a whole series of sectoral rates of profit becomes useful when introducing natural rates of profit themselves.

which is precisely the same condition as that required for matrix $\mathbf{H}c_i$, and therefore for the series expansion of matrix $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ itself, to converge.

Aggregate quantities are the sum of the $\mathbf{q}^{(i)}$'s, i.e.:

$$\begin{aligned}
\sum_{i=1}^{m}\mathbf{q}^{(i)} &= \sum_{i=1}^{m}(\mathbf{I}-\mathbf{H}c_i)^{-1}(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= \sum_{i=1}^{m}\left(\mathbf{I}+\mathbf{H}c_i+(\mathbf{H}c_i)^2+\ldots\right)(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= (\mathbf{I}+\mathbf{H})\sum_{i=1}^{m}\mathbf{x}^{(i)}+\mathbf{H}\sum_{i=1}^{m}c_i\left(\mathbf{I}+\mathbf{H}c_i+(\mathbf{H}c_i)^2+(\mathbf{H}c_i)^3+\ldots\right)(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= (\mathbf{I}+\mathbf{H})\mathbf{x}+\mathbf{H}(\mathbf{I}+\mathbf{H})\widehat{\mathbf{c}}\mathbf{x}+\mathbf{H}^2\sum_{i=1}^{m}c_i^2\left(\mathbf{I}+\mathbf{H}c_i+(\mathbf{H}c_i)^2+\ldots\right)(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= (\mathbf{I}+\mathbf{H})\mathbf{x}+(\mathbf{I}+\mathbf{H})\mathbf{H}\widehat{\mathbf{c}}\mathbf{x}+(\mathbf{I}+\mathbf{H})\mathbf{H}^2\widehat{\mathbf{c}}^2\mathbf{x}+\mathbf{H}^3\sum_{i=1}^{m}c_i^3\left(\mathbf{I}+\mathbf{H}c_i+\ldots\right)(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= (\mathbf{I}+\mathbf{H})(\mathbf{I}+(\mathbf{H}\widehat{\mathbf{c}})+(\mathbf{H}\widehat{\mathbf{c}})^2+(\mathbf{H}\widehat{\mathbf{c}})^3+\ldots)\mathbf{x}
\end{aligned} \tag{A.5}$$

If conditions (A.4) are satisfied, this infinite series converges, and therefore the quantity system, in the aggregate, can be written as:

$$\sum_{i=1}^{m}\mathbf{q}^{(i)} = \mathbf{q} = (\mathbf{I}+\mathbf{H})(\mathbf{I}-\mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} \tag{A.6}$$

Notice now that $\mathbf{q}^{(i)}$ $(i = 1, 2, \ldots, m)$ can be written as:

$$\mathbf{q}^{(i)} = (\mathbf{I}-\mathbf{H}c_i)^{-1}(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} \equiv (\mathbf{I}+\mathbf{H})(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \tag{A.7}$$

Since matrices $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ and $(\mathbf{I}+\mathbf{H})$ have the same eigenvectors, we can exploit the rule according to which if two matrices $\mathbf{X}$ and $\mathbf{Y}$ have the same eigenvectors, then $\mathbf{X} = \mathbf{Y} = \mathbf{YX}$ holds.

to see that $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ and $(\mathbf{I}+\mathbf{H})$ have the same eigenvectors, notice that, if

$$\mathbf{Ax} = \lambda_A\mathbf{x} \tag{A.8}$$

then

$$\mathbf{Ix}-\mathbf{Ax} = \mathbf{x}-\lambda_A\mathbf{x}$$

i.e.

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = (1 - \lambda_A)\mathbf{x}$$

Since if $\mathbf{Y}\mathbf{x} = \lambda_Y \mathbf{x}$ then $\mathbf{Y}^k\mathbf{x} = \lambda_Y^k \mathbf{x}$, then the latter expression can be written as

$$(\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} = \frac{1}{1 - \lambda_A}\mathbf{x} \tag{A.9}$$

and therefore

$$\mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} = \frac{1}{1 - \lambda_A}\mathbf{A}\mathbf{x}$$

which, by using expression (A.8) and the definition of $\mathbf{H}$, can be written as

$$\mathbf{H}\mathbf{x} = \lambda_H \mathbf{x}$$

where $\lambda_H = \lambda_A/(1 - \lambda_A)$. Therefore, it is shown that matrices $\mathbf{A}$ and $\mathbf{H}$ have the same eigenvectors. We can now write

$$\mathbf{H}c_i\mathbf{x} = \lambda_H c_i \mathbf{x}$$

and hence

$$(\mathbf{I} - \mathbf{H}c_i)\mathbf{x} = (1 - \lambda_H c_i)\mathbf{x}$$

and therefore

$$(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x} = \frac{1}{1 - \lambda_H c_i}\mathbf{x}$$

Moreover, since $(\mathbf{I} - \mathbf{A})^{-1} = (\mathbf{I} + \mathbf{H})$, expression (A.9) can be written as:

$$(\mathbf{I} + \mathbf{H})\mathbf{x} = \frac{1}{1 - \lambda_A}\mathbf{x}$$

We have therefore shown that matrices $(\mathbf{I}+\mathbf{H})$ and $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ have the same eigenvectors, and therefore that equivalence (A.7) holds.

Now, the expression for $\mathbf{s}^{(i)}$ can be written as:

$$\mathbf{s}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} = \mathbf{A}(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{H}(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{M}^{(i)}\mathbf{x}^{(i)} \tag{A.10}$$

For Perron-Frobenius theorems, matrices $\mathbf{M}^{(i)}$ $(i = 1, 2, \ldots, m)$ are non-negative provided that

$$c_i^{max} < \frac{1}{\lambda_H^{max}} \tag{A.11}$$

The aggregate vector of intermediate means of production, $\mathbf{s}$, is the sum of all the $\mathbf{s}^{(i)}$, i.e.:

$$\begin{aligned}
\mathbf{s} &= \sum_{i=1}^{m} \mathbf{s}^{(i)} = \sum_{i=1}^{m} \mathbf{H}(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \sum_{i=1}^{m} \mathbf{H}(\mathbf{I} + \mathbf{H}c_i + (\mathbf{H}c_i)^2 + \ldots)\mathbf{x}^{(i)} = \\
&= \sum_{i=1}^{m} (\mathbf{H} + \mathbf{H}^2 c_i + \mathbf{H}^3 c_i^2 + \ldots)\mathbf{x}^{(i)} = \sum_{i=1}^{m} (\mathbf{H}\mathbf{x}^{(i)} + \mathbf{H}^2 c_i \mathbf{x}^{(i)} + \mathbf{H}^3 c_i^2 \mathbf{x}^{(i)} + \ldots) = \\
&= \mathbf{H} \sum_{i=1}^{m} \mathbf{x}^{(i)} + \mathbf{H}^2 \sum_{i=1}^{m} c_i \mathbf{x}^{(i)} + \mathbf{H}^3 \sum_{i=1}^{m} c_i^2 \mathbf{x}^{(i)} + \ldots = \mathbf{H}\mathbf{x} + \mathbf{H}^2 \widehat{\mathbf{c}}\mathbf{x} + \mathbf{H}^3 \widehat{\mathbf{c}}^2 \mathbf{x} + \ldots = \\
&= \mathbf{H}(\mathbf{I} + \mathbf{H}\widehat{\mathbf{c}} + (\mathbf{H}\widehat{\mathbf{c}})^2 + \ldots) = \mathbf{H}(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} = \mathbf{M}\mathbf{x}
\end{aligned} \tag{A.12}$$

Hence, $\mathbf{s} = \mathbf{M}\mathbf{x}$. But $\mathbf{s}$ can also be written as:

$$\sum_{i=1} \mathbf{s}^{(i)} = \sum_{i=1}^{m} \mathbf{M}^{(i)}\mathbf{x}^{(i)} = \sum_{i=1}^{m} \mathbf{m}_i^* x_i = \overline{\mathbf{M}}\mathbf{x} \tag{A.13}$$

Comparing expressions (A.12) and (A.13), we can thus conclude that:

$$\mathbf{M}\mathbf{x} \cong \overline{\mathbf{M}}\mathbf{x}$$

By proceeding in the same way for sectoral and aggregate vertically hyper-integrated labour requirements too, we can conclude that

$$\mathbf{z}^T \mathbf{x} \cong \overline{\mathbf{z}}^T \mathbf{x}$$

If conditions (A.11) are satisfied, i.e. if all matrices $\mathbf{M}^{(i)}$, $i = 1, 2, \ldots, m$, are non-negative, then matrix $\overline{\mathbf{M}}$ is non-negative too.

## A.2 Reformulation of demand for capital goods

The total quantities of capital goods produced in one period is given by total quantities $\mathbf{q}^{(i)}$ less final demand $\mathbf{x}^{(i)}$. We can call this difference $\mathbf{q_k}^{(i)}$.

The total quantities produced in the $i$-th vertically hyper-integrated sector are:

$$\begin{aligned}
\mathbf{q}^{(i)} &= (\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \\
&= (\mathbf{I} + \mathbf{H})\left(\mathbf{I} + \mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\right)\mathbf{x}^{(i)} = \\
&= \mathbf{x}^{(i)} + \mathbf{H}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}(\mathbf{H}c_i)(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}
\end{aligned} \tag{A.14}$$

i.e. the sum of final demand for consumption good $i$ ($\mathbf{x}^{(i)}$), vertically integrated productive capacity for consumption good $i$ ($\mathbf{H}\mathbf{x}^{(i)}$), new investment ($\mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}$) and vertically integrated productive capacity for new investments ($\mathbf{H}(\mathbf{H}c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}$).

Therefore:

$$\begin{aligned}\mathbf{q_k}^{(i)} &= \mathbf{H}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}(\mathbf{H}c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \\ &= \mathbf{H}\left(\mathbf{I} + c_i(\mathbf{I}-\mathbf{H}c_i)^{-1} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\right)\mathbf{x}^{(i)} = \\ &= \mathbf{H}(1+c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}\end{aligned} \tag{A.15}$$

Following Pasinetti (1981), $\mathbf{q_k}^{(i)}$ can also be written as:

$$\mathbf{q_k}^{(i)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{A}\mathbf{q_k}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \tag{A.16}$$

i.e. direct productive capacity for consumption good $i$ plus direct productive capacity for $\mathbf{q_k}^{(i)}$ plus new investment. Using the last equality of (A.15) this expression can be written as:

$$\begin{aligned}\mathbf{q_k}^{(i)} &= \mathbf{A}\mathbf{x}^{(i)} + \mathbf{A}\mathbf{H}(1+c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \\ &= \mathbf{A}\mathbf{x}^{(i)} + \mathbf{A}\mathbf{H}(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{A}\mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \\ &= \mathbf{H}\mathbf{x}^{(i)} + \mathbf{H}(\mathbf{H}c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}\end{aligned} \tag{A.17}$$

which is precisely the first line of (A.15). Hence (A.15) and (A.16) are equivalent.

## A.3 The price of the units of vertically hyper-integrated productive capacity

Show that expressions (5.67) and (5.62) are equivalent, i.e. that

$$p_{k_i} = \overline{w}\overline{\mathbf{z}}^T\mathbf{m}_i^* + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}})\mathbf{m}_i^* \equiv \overline{w}\mathbf{z}^{(i)}\mathbf{m}_i^* + \mathbf{p}^T\mathbf{M}^{(i)}(\pi-c_i)\mathbf{m}_i^* \tag{A.18}$$

The first equality can be written as:

$$\begin{aligned}p_{k_i} =& \overline{w}\overline{\mathbf{z}}^T\mathbf{m}_i^* + \overline{w}\mathbf{z}^{(i)}\mathbf{m}_i^* - \overline{w}\mathbf{z}^{(i)}\mathbf{m}_i^* + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}})\mathbf{m}_i^* + \\ &+ \mathbf{p}^T\mathbf{M}^{(i)}(\pi-c_i)\mathbf{m}_i^* - \mathbf{p}^T\mathbf{M}^{(i)}(\pi-c_i)\mathbf{m}_i^*\end{aligned}$$

i.e. as:

$$\begin{aligned}p_{k_i} =& \overline{w}\mathbf{z}^{(i)}\mathbf{m}_i^* + \mathbf{p}^T\mathbf{M}^{(i)}(\pi-c_i)\mathbf{m}_i^* + \\ &+ \overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}})\mathbf{m}_i^* - \mathbf{p}^T\mathbf{M}^{(i)}(\pi-c_i)\mathbf{m}_i^*\end{aligned}$$

We therefore want to show that

$$\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* - \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^* = 0 \tag{A.19}$$

By exploiting equivalence (5.58), expression (A.19) can be written as:

$$\begin{aligned}&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \overline{w}\overline{\mathbf{z}}^T\left(\mathbf{I} - \overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* +\\ &- \overline{w}\mathbf{z}^{(i)T}\left(\mathbf{I} - \mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^*\end{aligned}$$

Since, given two matrices $\mathbf{F}$ and $\mathbf{G}$, it holds that $\mathbf{F}^{-1}\mathbf{G} \equiv (\mathbf{G}^{-1}\mathbf{F})^{-1}$, the latter can be written as:

$$\begin{aligned}&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \overline{w}\overline{\mathbf{z}}^T\left(\left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}\left(\mathbf{I} - \overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}\right)^{-1}\mathbf{m}_i^* +\\ &- \overline{w}\mathbf{z}^{(i)T}\left(\left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}\left(\mathbf{I} - \mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}\right)^{-1}\mathbf{m}_i^* =\\ =&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \overline{w}\overline{\mathbf{z}}^T\left(\left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1} - \mathbf{I}\right)^{-1}\mathbf{m}_i^* +\\ &- \overline{w}\mathbf{z}^{(i)T}\left(\left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1} - \mathbf{I}\right)^{-1}\mathbf{m}_i^* =\\ =&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* - \overline{w}\overline{\mathbf{z}}^T\left(\mathbf{I} - \left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}\right)^{-1}\mathbf{m}_i^* +\\ &+ \overline{w}\mathbf{z}^{(i)T}\left(\mathbf{I} - \left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}\right)^{-1}\mathbf{m}_i^*\end{aligned}$$

which, by defining $\left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1} \equiv \mathbf{F}$ and $\left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1} \equiv \mathbf{G}$, becomes

$$\begin{aligned}&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* - \overline{w}\overline{\mathbf{z}}^T(\mathbf{I} - \mathbf{G})^{-1}\mathbf{m}_i^* + \overline{w}\mathbf{z}^{(i)T}(\mathbf{I} - \mathbf{F})^{-1}\mathbf{m}_i^* =\\ =&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* - \overline{w}\overline{\mathbf{z}}^T\left(\mathbf{I} + \mathbf{G}(\mathbf{I} - \mathbf{G})^{-1}\right)\mathbf{m}_i^* + \overline{w}\mathbf{z}^{(i)T}\left(\mathbf{I} + \mathbf{F}(\mathbf{I} - \mathbf{F})^{-1}\right)\mathbf{m}_i^* =\\ =&\overline{w}\mathbf{z}^{(i)T}\mathbf{F}(\mathbf{I} - \mathbf{F})^{-1}\mathbf{m}_i^* - \overline{w}\overline{\mathbf{z}}^T\mathbf{G}(\mathbf{I} - \mathbf{G})^{-1}\mathbf{m}_i^*\end{aligned} \tag{A.20}$$

By substituting back into (A.20) the definitions of $\mathbf{F}$ and $\mathbf{G}$, we get

$$\begin{aligned}&\overline{w}\mathbf{z}^{(i)T}\left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}(\mathbf{I} - \left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1})^{-1}\mathbf{m}_i^* +\\ &- \overline{w}\overline{\mathbf{z}}^T\left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}(\mathbf{I} - \left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1})^{-1}\mathbf{m}_i^*\end{aligned}$$

which can be written as

$$\overline{w}\mathbf{z}^T \left(\mathbf{I} - \overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1} - \overline{w}\mathbf{z}^{(i)T} \left(\mathbf{I} - \mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}$$

which, again using equivalence (5.58), is finally shown to be equal to zero.

Going back to what hinted at in section 6 about the reason because of which the labour costs and the profit component can differ in (5.67) with respect to (5.62), we now have to derive expression (6.2):

$$c_i \lesseqgtr \frac{\mathbf{v}^T\mathbf{D}^{(i)}\widehat{\mathbf{c}}\mathbf{m}_i^*}{\mathbf{v}^T\mathbf{D}^{(i)}\mathbf{m}_i^*}, \qquad i = 1, 2, \ldots, m \tag{6.2}$$

Matrix $\mathbf{D}^{(i)} = [\mathbf{d}_j^{(i)}]$ is the first derivative of matrix $(\mathbf{I} - \mathbf{H}c_i)^{-1}$, that is to say

$$\mathbf{D}^{(i)} = \mathbf{H}(\mathbf{I} - (\mathbf{H}c_i)^2)^{-1}$$

which is non-negative provided that $c_i < 1/\lambda_H^{max}$, i.e. whenever $(\mathbf{I} - \mathbf{H}c_i)^{-1}$ is itself non-negative.

Since $\mathbf{z}^{(i)T} = \mathbf{v}^T(\mathbf{I} - \mathbf{H}c_i)^{-1}$ and $\overline{\mathbf{z}}^T = \mathbf{v}^T(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}$, variations in the rate(s) of change in final demand affect both of them only through the effect such variations have on matrices $(\mathbf{I} - \mathbf{H}c_i)^{-1}$ and $(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}$, respectively.[33] Moreover, we can say that:

$$\frac{\mathrm{dz_i^*}}{\mathrm{dc_i}} = \mathbf{v}^T\mathbf{d}_i^{(i)}, \quad \frac{\mathrm{dz_j^{(i)}}}{\mathrm{dc_i}} = \mathbf{v}^T\mathbf{d}_j^{(i)}, \qquad \forall i, j = 1, 2, \ldots, m \tag{A.21}$$

and moreover that:

$$z_j^* = z_j^{(i)} + \mathbf{v}^T\mathbf{d}_j^{(i)}(c_j - c_i), \qquad \forall i, j = 1, 2, \ldots, m \tag{A.22}$$

We can now look for the relation between labour cost component in the two equivalent expressions (5.67) and (5.62) and the growth differentials between the $m$ vertically hyper-integrated sectors composing the economic system. Such a difference is given by:

$$(\mathbf{z}^{(i)T} - \overline{\mathbf{z}}^T)\mathbf{m}_i^* = \left([z_1^{(i)}, \ldots, z_i^*, \ldots, z_m^{(i)}] - [z_1^*, \ldots, z_i^*, \ldots, z_m^*]\right)\mathbf{m}_i^*$$

which, by using expression (A.22) becomes:

$$(\mathbf{z}^{(i)T} - \overline{\mathbf{z}}^T)\mathbf{m}_i^* = \left(-[\mathbf{v}^T\mathbf{d}^{(i)}(c_1 - c_i), \ldots, 0, \ldots, \mathbf{v}^T\mathbf{d}^{(i)}(c_m - c_i)]\right)\mathbf{m}_i^*$$

[33] Recall, as a matter of notation, that $\mathbf{z}^{(i)T} = [z_1^{(i)}, \ldots, z_i^*, \ldots, z_m^{(i)}]$ and $\overline{\mathbf{z}}^T = [z_1^*, \ldots, z_i^*, \ldots, z_m^*]$.

or

$$(\mathbf{z}^{(i)T} - \overline{\mathbf{z}}^T)\mathbf{m}_i^* = \mathbf{v}^T \left( \mathbf{D}^{(i)} \mathbf{m}_i^* c_i - \mathbf{D}^{(i)} \widehat{\mathbf{c}} \mathbf{m}_i^* \right)$$

Therefore, we might conclude that

$$(\mathbf{z}^{(i)T} - \overline{\mathbf{z}}^T)\mathbf{m}_i^* \lesseqgtr 0$$

according to whether

$$c_i \lesseqgtr \frac{\mathbf{v}^T \mathbf{D}^{(i)} \widehat{\mathbf{c}} \mathbf{m}_i^*}{\mathbf{v}^T \mathbf{D}^{(i)} \mathbf{m}_i^*} \tag{6.2}$$

i.e. according to whether $c_i$ is greater than, equal to o smaller than a particular weighted average of *all* the rates of growth of demand for consumption commodities, and which is precisely what we wanted to show.

To conclude, when $c_i$ is smaller than this weighted average, expression (5.67) does over-estimate the labour cost associated to the production of one unit of vertically hyper-integrated productive capacity for vertically hyper-integrated sector $i$ — in terms of labour, it is less costly to produce each component in the sector producing the corresponding commodity as a consumption good than in sector $i$ itself; this greater labour cost is however compensated by a smaller — to the same extent — profit component, since clearly the differential with respect to the various rates of growth of demand is smaller than the specific differential corresponding to production in vertically hyper-integrated sector $i$, whose rate of growth is smaller. The sum of these two components is anyway exactly equal to that coming from expression (5.62). As a consequence, when we simply need the total unitary price of vertically hyper-integrated units of productive capacity e can use either formulation (5.67) or (5.62); when our aim is that of analysing the two components independently, the correct formulation to be used is (5.62) and, additionally, the complete set of intermediate commodity prices (5.60), together with solutions (6.4).

## A.4 The more complex formulation

We will consider here the more complex case, where both consumption goods and capital goods are produced by means of labour and capital goods (Pasinetti 1981, chapter II, section 7). Pasinetti's (1981) original physical quantity system, in this more complex case, is:[34]

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\mathbf{I} & \mathbf{I}-\widehat{\boldsymbol{\gamma}} & -\mathbf{a}_{k_i n} \\ -\mathbf{a}^T_{ni} & -\mathbf{a}^T_{nk_i} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{A.23}$$

The simplifying assumption made here is that each vertically integrated sector $i$ is made up by only two industries: one producing the final commodity $i$ and the other producing the homogeneous capital good $k_i$ used by both of them. Such capital goods are sector-specific — i.e. different from sector to sector — commodities, measured in units of direct productive capacity for the final commodity industry. When this particular unit of measurement is used, the production of one unit of the final commodity requires, by definition, one unit of the capital good $k_i$.

In order to understand the meaning of the $\gamma_i$s, as described by Pasinetti (1981, p. 43), let us go back to ordinary units, calling $\alpha_i$ the number of units of commodity $k_i$ necessary for the production of one unit of commodity $i$, and $\gamma_i$ the number of units of commodity $i$ to be used for the production of one unit of commodity $k_i$ itself. Then, a unit of productive capacity for the productive capacity industry — i.e. the number of units of the capital goods necessary for the production of one unit of productive capacity for the consumption good — is made up by $\alpha_i\gamma_i$ ordinary units of the capital good, or by $\gamma_i$ units of productive capacity for the consumption good.

If the total quantity of units of capital good available is $K_i$, it can be used either for producing productive capacity for the consumption good, hence obtaining $1/\alpha_i$ such units, or for producing productive capacity for productive capacity, hence obtaining $1/\gamma_i\alpha_i$ such units. The ratio of these two quantities is

$$\frac{1/\alpha_i}{1/\gamma_i\alpha_i} = \gamma_i$$

Therefore, the $\gamma_i$'s express the number of ordinary units of commodity $k_i$ necessary for its own reproduction, the number of units of productive capacity for the consumption goods necessary for the production of one such unit of productive capacity, and the ratio of the total stock of capital goods expressed in terms of units of productive capacity for the consumption good, to the stock of capital goods expressed in terms of units of productive capacity for the productive capacity itself.

From now on, for the whole section, when talking about quantity of capital goods, we will always be using units of productive capacity for the consumption good as the unit of measurement.

[34]Since we are dealing with circulating capital only, we will set here $T_i = T_{k_i} = 1$, which means that the depreciation rate is equal to 1 in all industries and hence in all sectors.

The total quantity of capital goods to be produced in each period is given by the sum of the number of units of final commodity $i$, the number of units of productive capacity demanded as net investment ($a_{k_i n} x_n$, where $a_{k_i n}$ is the per-capita demand of units of productive capacity as new investments) and the number of units of productive capacity that have to be used up and therefore replaced ($\gamma_i x_{k_i}$).

Vector $\mathbf{a}_{ni}^T$ is the vector of unitary direct labour requirements for the final commodities, and $\mathbf{a}_{nk_i}^T$ is the vector of direct labour requirements per unit of productive capacity.

Written as an eigenvalue problem, system (A.23) is:[35]

$$\begin{cases} (\lambda_x^* \overline{\mathbf{I}} - \overline{\mathbf{A}}_x)\overline{\mathbf{x}} = \overline{\mathbf{0}} \\ \lambda_x^* = 1 \\ \lambda_x^* = \lambda_x^{max} \end{cases} \tag{A.25}$$

$\overline{\mathbf{x}}$, i.e. the solution for physical quantities, is the right-hand eigenvector of the non-negative matrix $\overline{\mathbf{A}}$ associated to a unitary eigenvalue. Therefore, in order for the eigensystem to have a solution, $\lambda_x^* = 1$ must be an eigenvalue of matrix $\overline{\mathbf{A}}$; moreover, in order for such solution to be real and non-negative, such eigenvalue must be the maximum one.

The characteristic equation associated to system (A.25) can be written as:

$$|\mathbf{O} - \lambda_x^* \mathbf{I}||\widehat{\boldsymbol{\gamma}} - \lambda_x^* \mathbf{I}| \left( -\lambda_x^* - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T \end{bmatrix} \begin{bmatrix} -\lambda_x^* \mathbf{I} & \mathbf{O} \\ \mathbf{I} & \widehat{\boldsymbol{\gamma}} - \lambda_x^* \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} \\ \mathbf{a}_{k_i n} \end{bmatrix} \right) = 0 \tag{A.26}$$

We notice first that the eigenvalues of matrix $\overline{\mathbf{A}}_x$ will be different from zero — those of matrix $\mathbf{O}$ — and from $\gamma_i$, $(i = 1, 2, \ldots, m)$ — those of matrix $\widehat{\boldsymbol{\gamma}}$ — otherwise matrices $(\mathbf{O} - \lambda_x^* \mathbf{I})$ and $(\widehat{\boldsymbol{\gamma}} - \lambda_x^* \mathbf{I})$ would not be invertible.

The $m+1$ eigenvalues of matrix $\overline{\mathbf{A}}_x$ are obtained as the solutions to

$$\mathbf{a}_{ni}^T \frac{1}{\lambda_x^*} \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \frac{1}{\lambda_x^*} (\lambda_x^* \mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1} \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T (\lambda_x^* \mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1} \mathbf{a}_{k_i n} = \lambda_x^* \tag{A.27}$$

which in turn tells us that the $\gamma_i$s must be smaller than the maximum eigenvalue of $\overline{\mathbf{A}}_x$.

Since we want one solution to be $\lambda_x^* = 1$, the condition for this to be true is that:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T (\mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1} \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T (\mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1} \mathbf{a}_{k_i n} = 1 \tag{A.28}$$

[35] Where:

$$\overline{\mathbf{A}}_x = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} \\ \mathbf{I} & \widehat{\boldsymbol{\gamma}} & \mathbf{a}_{k_i n} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T & 0 \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} \tag{A.24}$$

which is precisely the macroeconomic condition found by Pasinetti (1981).

If such condition holds, then $\lambda_x^* = 1$ also is the maximum solution: since all the terms in equation (A.27) are decreasing functions of $\lambda_x^*$, the presence of an eigenvalue greater than one would contradict (A.28).

Being the maximum eigenvalue of matrix $\overline{\mathbf{A}}_x$ equal to 1, the above-mentioned conditions on the value of the $\gamma_i$'s reduces to $\gamma_i < 1, \forall i$, which is a *viability condition* for the physical quantity system: the production of one unit of productive capacity cannot require more than one unit of productive capacity itself. If this condition were not accomplished, the economic system would not be viable.

The eigenvector associated to $\lambda_x^* = 1$ is therefore the solution for physical quantities, completely determined once we set $x_n = \overline{x}_n$, thus obtaining:

$$\begin{cases} \mathbf{x} = \mathbf{a}_{in}\overline{x}_n \\ \mathbf{x}_k = (\mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1}(\mathbf{a}_{in} + \mathbf{a}_{k_i n})\overline{x}_n \\ w = \overline{w} \end{cases} \tag{A.29}$$

As to the price system, it is given by:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p_k}^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -(\mathbf{I} + \widehat{\boldsymbol{\pi}}) & \mathbf{I} - \widehat{\mathbf{B}}_k & \widehat{\boldsymbol{\Gamma}}_{in}\mathbf{a}_{in} - \widehat{\boldsymbol{\Gamma}}_{k_i n}\mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{bmatrix} = \overline{\mathbf{0}}^T \tag{A.30}$$

where:

$$\begin{aligned} \widehat{\mathbf{B}} &= \widehat{\boldsymbol{\gamma}}(\mathbf{I} + \widehat{\boldsymbol{\pi}}) \\ \widehat{\mathbf{B}}_k &= \widehat{\boldsymbol{\gamma}}(\mathbf{I} + \widehat{\boldsymbol{\pi}}_k) \\ \widehat{\boldsymbol{\Gamma}}_{in} &= (\widehat{\boldsymbol{\pi}} + \widehat{\mathbf{B}}_k - \widehat{\mathbf{B}})(\mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1} \\ \widehat{\boldsymbol{\Gamma}}_{k_i n} &= (\mathbf{I} - \widehat{\mathbf{B}}_k)(\mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1} \end{aligned}$$

As an eigenproblem, system (A.30) becomes:[36]

$$\begin{cases} \overline{\mathbf{p}}^T(\lambda_p \overline{\mathbf{I}} - \overline{\mathbf{A}}_p) = \overline{\mathbf{0}}^T \\ \lambda_p^* = 1 \end{cases} \tag{A.32}$$

[36]Where:

$$\overline{\mathbf{A}}_p = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} \\ \mathbf{I} + \widehat{\boldsymbol{\pi}} & \widehat{\mathbf{B}}_k & -\widehat{\boldsymbol{\Gamma}}_{in}\mathbf{a}_{in} + \widehat{\boldsymbol{\Gamma}}_{k_i n}\mathbf{a}_{k_i n} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T & 0 \end{bmatrix} \tag{A.31}$$

with characteristic equation:

$$|\mathbf{O}-\lambda_p\mathbf{I}||\widehat{\mathbf{B}}_k-\lambda_p\mathbf{I}|\left(\begin{bmatrix}\mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T\end{bmatrix}\begin{bmatrix}-\lambda_p^*\mathbf{I} & \mathbf{O}\\ \mathbf{I}+\widehat{\boldsymbol{\pi}} & \widehat{\mathbf{B}}_k-\lambda_p^*\mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{a}_{in}\\ -\widehat{\boldsymbol{\gamma}}_{in}\mathbf{a}_{in}+\widehat{\boldsymbol{\gamma}}_{k_in}\mathbf{a}_{k_in}\end{bmatrix}\right)=0 \tag{A.33}$$

Matrix $\overline{\mathbf{A}}_p$ can be either negative or non-negative, depending on the sign of the last element of the second row: it is non-negative as long as total profits do not exceed the total value of new investments ($\mathbf{p}_k^T\mathbf{a}_{k_in}x_n$). Anyway, this is not necessarily true, and therefore, in solving this eigenproblem, we are not going to use Perron Frobenius theorems. We will simply find out the condition for $\lambda_p^*=1$ to be an eigenvalue of $\overline{\mathbf{A}}_p$, compute the associated eigenvector, and set the conditions for it to be real and non-negative.

We know that the eigenvalues of matrix $\overline{\mathbf{A}}_p$ will be different from zero — the eigenvalues of matrix $\mathbf{O}$ — and from $\gamma_i(1+\pi_{k_i})$, $(i=1,2,\ldots,m)$ — the eigenvalues of matrix $\widehat{\mathbf{B}}_k$, or matrices $(-\lambda_p^*)$ and $(\lambda_p^*\mathbf{I}-\widehat{\mathbf{B}}_k)$ would not be invertible.

The $m+1$ eigenvalues of matrix $\overline{\mathbf{A}}_p$ are thus the solutions of:

$$\begin{aligned}&\frac{1}{\lambda_p^*}\mathbf{a}_{ni}^T\mathbf{a}_{in}+\mathbf{a}_{nk_i}^T(\lambda_p^*\mathbf{I}-\widehat{\mathbf{B}}_k)^{-1}\left(\frac{1}{\lambda_p^*}(\mathbf{I}+\widehat{\boldsymbol{\pi}})(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}+\widehat{\mathbf{B}}-\widehat{\mathbf{B}}_k-\widehat{\boldsymbol{\pi}}\right)(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}\mathbf{a}_{in}+\\&+\mathbf{a}_{nk_i}^T(\lambda_p^*\mathbf{I}-\widehat{\mathbf{B}}_k)^{-1}(\mathbf{I}-\widehat{\mathbf{B}}_k)(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}\mathbf{a}_{k_in}=\lambda_p^*\end{aligned} \tag{A.34}$$

which, when $\lambda_p^*=1$, reduces to:

$$\mathbf{a}_{ni}^T\mathbf{a}_{in}+\mathbf{a}_{nk_i}^T(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}\mathbf{a}_{in}+\mathbf{a}_{nk_i}^T(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}\mathbf{a}_{k_in}=1 \tag{A.35}$$

which is the same condition as the one found above for the quantity system.

By fixing $w=\overline{w}$, the eigenvector associated to $\lambda_p^*$ is:

$$\begin{cases}\mathbf{p}^T=\overline{w}\left(\mathbf{a}_{ni}^T+\mathbf{a}_{nk_i}^T(\mathbf{I}+\widehat{\boldsymbol{\pi}})(\mathbf{I}-\widehat{\boldsymbol{\gamma}}(\mathbf{I}+\widehat{\boldsymbol{\pi}}_k))^{-1}\right)\\ \mathbf{p}_k^T=\overline{w}\mathbf{a}_{nk_i}^T(\mathbf{I}-\widehat{\boldsymbol{\gamma}}(\mathbf{I}+\widehat{\boldsymbol{\pi}}_k))^{-1}\\ w=\overline{w}\end{cases} \tag{A.36}$$

which is real and non-negative as long as:

$$\pi_{k_i}<\frac{1-\gamma_i}{\gamma_i},\quad i=1,2,\ldots,m \tag{A.37}$$

Expression (A.37) therefore is a set of sectoral *viability conditions* for the price system, telling us the maximum rate of profit which cannot be exceeded if we want prices to be non-negative. Notice that such a viability condition involves the rates of profit of the industries producing intermediate commodities, and not producing consumption goods (see Pasinetti 1981, p. 45, with $T_{k_i}=1$).

# Structural Change and Economic Growth: Production in the Long Run — A generalisation in terms of vertically hyper-integrated sectors

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**Abstract** Pasinetti's (1981) *Structural Change and Economic Growth* provides a complete and far reaching theoretical framework for the study of structural change, and therefore of economic development, rooted in in the Classical-Sraffian tradition.

Some attempts have been made, both in the '80s — for instance Siniscalco (1982) and Momigliano & Siniscalco (1986) — and more recently — e.g. Montresor & Vittucci Marzetti (2007a) and Montresor & Vittucci Marzetti (2008) — to use this framework for empirical purposes. However, all these attempts are based on Pasinetti's (1973) paper, i.e. on vertically integrated analysis. It is my contention that, as a consequence, they failed to recognise, and therefore to take advantage of, the main analytical feature of the 1981 book, namely vertical *hyper-integration*.

Actually, when trying to overcome the simplifying assumptions made by Pasinetti (1981) as regards the description of the technique, the starting point should be Pasinetti (1988), and not Pasinetti (1973), the latter being an intermediate step leading to the former.

After having highlighted the key differences between Pasinetti (1973) and Pasinetti (1988) — in order to show Pasinetti's (1981) vertically *hyper*-integrated character — and having generalised — by reintroducing inter-industry relations and allowing for more complex dynamics of economic magnitudes — the analytical framework provided by Pasinetti (1981) itself as to *production in the short run* (see Garbellini 2010b), the aim of the present paper is that of facing the issue of *production in the long run*, i.e. of extending the above mentioned generalisation to the 'general multisector dynamic model' (Pasinetti 1981, chapter V) presented by Pasinetti in his 1981 book.

This conceptual clarification and analytical generalisation is intended to be the first step of a line of research aiming at using, and extending, the present framework to perform empirical analyses and study the behaviour of actual economic systems.

**Keywords** Natural system, vertically integrated sectors, vertically hyper-integrated sectors, functional income distribution, natural rates of profit, natural prices.

**JEL classification** B51,L16,O41

## 1 Introduction

In his 1981 book, Pasinetti goes into many topics concerning economic *theory* — e.g. the accumulation of capital — and *reality* — e.g. international relations. Anyway, the most complete and general formulation of the quantity and price systems used as a starting point for the development of the whole framework is given in Pasinetti (1988), where the notion of vertically hyper-integrated sector — or growing subsystem — is rigorously introduced.

After having clarified the vertically hyper-integrated character of Pasinetti's (1981) framework, and having restated and generalised the quantity and price systems, their solutions, and the equilibrium conditions characterising production in the *short run* (see Garbellini 2010b), this paper aims at doing the same with the topics touched upon in the second part of the book, i.e. that devoted to production in the *long run*.

More specifically — after providing some basic notation in section 2 and reassessing production in the short run in section 3 — section 4 sets up the general multi-sector dynamic model: the initial conditions and the laws of motion are stated, and therefore the 'dynamic' equilibrium conditions are derived.

Then, section 5 touches upon the topic of *changes in labour productivity*, singling out how the change in (total) labour productivity in each vertically hyper-integrated sector $i$ ($i = 1, 2, \ldots, m$) is the (weighted) average of the rate of change of *direct*, *indirect* and *hyper-indirect* labour productivity — or, alternatively, of the rate of change of direct and indirect labour productivity for consumption commodity $i$ and direct and indirect labour productivity for the corresponding *additional* productive capacity. Some reflections are made on the usefulness of this analytical decomposition for empirical purposes.

Section 6 is a note on the degrees of freedom left open when the price system is considered *through time*, and the implications that they have on the choice of the *numéraire*.

Section 7 then goes through the structural dynamics of physical quantities and commodity prices, stressing how the whole structure of the economic system — looking both at the physical and at the value side — is continuously changing through time, due to the presence of non-uniform (among vertically hyper-integrated sectors) and of non-steady rates of growth of sectoral demand for consumption commodities and labour productivity — and to their intermingled dynamics.

Section 8 first recalls the difference between capital intensity — as expressed by the capital/output ratios(s) — and degree of mechanisation — as expressed by the capital/labour ratio(s) — and, moreover, between the sectoral and the aggregate expressions for such ratios. Then, the dynamics of both the sectoral and the aggregate ratios is analysed, in order to single out the corresponding determinants.

Section 9 introduces the 'natural' economic system; first, the particular theory of income distribution leading to it is briefly exposed, the 'natural' rate of profits are defined, and the 'natural' price system(s) — together with their properties and features — are stated (section 9.1). Then, the particular configuration of *sectoral* capital/output and capital/labour ratios *within the 'natural' economic system* is analysed, in order to single out the determinants of their dynamics through time as opposed to those characterising them when prices are not the 'natural' ones (section 9.2). Third, the concepts of 'standard rate of growth of productivity' — and hence of 'dynamic standard commodity' — are introduced (section 9.3).

Then, section 10 deals with the issue of the choice of the *numéraire* — with special reference to a conventional unit of account, thereby reaching a definition, to be used in the last section, of the general rate of price inflation — for the price system, again looking at the implications of such a choice on the closure of the two degrees of freedom left open.

Finally, section 11 closes the essay, by introducing the concept of 'natural' rate of interest, and hence extending the principle of labour income and value distribution also to those exchanges that shift purchasing power through time.

Some final remarks are provided in section 12.

## 2 Basic notation

Consider an economic system in which $m$ commodities, denoted by subscript $i$ ($i = 1, 2, \ldots, m$) are produced. Such commodities can be used *either* as (pure) consumption goods *or* as intermediate commodities, *or* both.

Moreover, make the simplifying assumption that those commodities used as means of production are completely used up in each period, and therefore have to be replaced entirely.[1]

The economic system can be described by:

| | | | |
|---|---|---|---|
| $\mathbf{q}$ | $=$ | $[q_i]$: | vector of total quantities; |
| $\mathbf{x}$ | $=$ | $[x_i]$: | vector of final demand for consumption goods; |
| $\mathbf{j}$ | $=$ | $[j_i]$: | vector of final demand for investment goods; |
| $\mathbf{y}$ | $=$ | $[y_i]$: | vector of final demand, with $y_i = x_i + j_i$, $i = 1, 2, \ldots, m$; |
| $\mathbf{A}$ | $=$ | $[a_{ij}]$: | matrix of inter-industry coefficients; |
| $\mathbf{a}_{ni}$ | $=$ | $[a_{ni}]$: | vector of direct labour requirements; |
| $\mathbf{a}_{in}$ | $=$ | $[a_{in}]$: | vector of demand coefficients for consumption goods: $x_i = a_{in} x_n$; |
| $\mathbf{a}_{k_i n}$ | $=$ | $[a_{k_i n}]$: | vector of demand coefficients for new investment: $j_i = a_{k_i n} x_n$; |
| $\mathbf{s}$ | $=$ | $[s_i]$: | vector of intermediate commodities necessary for the production of quantities $q_i$; |
| $\mathbf{p}$ | $=$ | $[p_i]$: | vector of commodity prices; |
| | | $x_n$: | total labour. |
| | | $g$: | rate of growth of population; |
| | | $r_i$: | rate of growth of per-capita (average) demand of commodity $i$ as a final good; |
| | | | $(i = 1, \ldots, m)$ |

[1] No treatment of fixed capital is made here. This simplification is intended to be a first step to be followed by a complete treatment of this issue too. However, since extending the description of the technology in use introduces many complications, I have decided to limit myself, for the time being, to consider circulating capital only.

and the derived magnitudes obtained in Garbellini (2010b) are:

$$
\begin{aligned}
&\mathbf{H} = \mathbf{A}(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{A}(\mathbf{I}+\mathbf{H})\\
&\mathbf{v}^T = \mathbf{a}_{ni}^T(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{a}_{ni}^T(\mathbf{I}+\mathbf{H})\\
&\mathbf{M} = \mathbf{H}(\mathbf{I}-\mathbf{H}\widehat{\mathbf{c}})^{-1}\\
&\mathbf{z}^T = \mathbf{v}^T(\mathbf{I}-\mathbf{H}\widehat{\mathbf{c}})^{-1}\\
&\mathbf{M}^{(i)} = \mathbf{H}(\mathbf{I}-\mathbf{H}c_i)^{-1} = [\mathbf{m}_1^{(i)}\ldots,\mathbf{m}_i^*,\ldots,\mathbf{m}_m^{(i)}], \qquad i=1,2,\ldots,m\\
&\mathbf{z}^{(i)T} = \mathbf{v}(\mathbf{I}-\mathbf{H}c_i)^{-1} = [z_1^{(i)},\ldots,z_i^*,\ldots,z_m^{(i)}], \qquad i=1,2,\ldots,m\\
&\mathbf{z}_k^{(i)T} = \mathbf{z}^{(i)T}\mathbf{M}^{(i)} = [z_{k_1}^{(i)},\ldots,z_{k_i}^*,\ldots,z_{k_m}^{(i)}]\\
&\overline{\mathbf{M}} = [\mathbf{m}_1^*,\ldots,\mathbf{m}_i^*,\ldots,\mathbf{m}_m^*]\\
&\overline{\mathbf{z}}^T = [z_1^*,\ldots,z_i^*,\ldots,z_m^*]\\
&\overline{\mathbf{z}}_k^T = \overline{\mathbf{z}}^T\overline{\mathbf{M}} = [z_{k_1},\ldots,z_{k_i},\ldots,z_{k_m}]\\
&\boldsymbol{\Phi}(\pi) = [\boldsymbol{\phi}_i(\pi)] = \big(\mathbf{I}-\overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}})\big)^{-1}\\
&\boldsymbol{\Phi}^{(i)}(\pi) = [\boldsymbol{\phi}_i^{(i)}(\pi)] = \Big(\mathbf{I}-\mathbf{M}^{(i)}(\pi-c_i)\Big)^{-1}\\
&\boldsymbol{\Phi}_k(\pi) = [\boldsymbol{\phi}_{k_i}(\pi)] = \big(\mathbf{I}-(\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}}\big)^{-1}\\
&\mathbf{D}^{(i)} = [\mathbf{d}_i^{(i)}] = \frac{\mathrm{d}}{\mathrm{dc_i}}(\mathbf{I}-\mathbf{H}c_i)^{-1} = \big(\mathbf{I}-(\mathbf{H}c_i)^2\big)^{-1}
\end{aligned}
$$

All throughout the paper, the following conventions will be observed:

- All vectors and matrices will be denoted by boldface symbols, while all scalar quantities by normal type ones;
- all matrices will be denoted by upper case letters, while all vectors by lower case ones;
- all vectors will be intended as column vectors; row vectors will be denoted by transposed vectors;
- a vector with a hat will denote a diagonal matrix with the element of the corresponding vector on the main diagonal.

## 3 Production in the short run: a reassessment

In order to be able to face the dynamic part of the framework, it is worth briefly recalling the quantity and price systems, their solutions, and the equilibrium conditions guaranteeing full employment — of the labour force and of productive capacity — and full expenditure of income.[2]

The quantity and price systems can be written, respectively, as:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\mathbf{I} & \mathbf{I} & -\widehat{\mathbf{c}}\mathbf{a}_{in} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{ni}^T\overline{\mathbf{M}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{3.1}$$

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -(\pi\mathbf{I}-\widehat{\mathbf{c}}) & \mathbf{I}-(\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}} & (\pi\mathbf{I}-\widehat{\mathbf{c}})\mathbf{a}_{in} \\ -\overline{\mathbf{z}}^T & -\overline{\mathbf{z}}^T\overline{\mathbf{M}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \tag{3.2}$$

their solutions being, respectively:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_k \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{in}\overline{x}_n \\ (\mathbf{I}+\widehat{\mathbf{c}})\mathbf{a}_{in}\overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{3.3}$$

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} = \begin{bmatrix} \overline{w}\left[\overline{\mathbf{z}}^T\left(\mathbf{I}-\overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}})\right)^{-1}\right] \\ \overline{w}\left[\overline{\mathbf{z}}^T\overline{\mathbf{M}}\left(\mathbf{I}-(\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}}\right)^{-1}\right] \\ \overline{w} \end{bmatrix}^T = \begin{bmatrix} \overline{w}\overline{\mathbf{z}}^T\mathbf{\Phi}(\pi) \\ \overline{w}\overline{\mathbf{z}}_k^T\mathbf{\Phi}_k(\pi) \\ \overline{w} \end{bmatrix}^T \tag{3.4}$$

Moreover, the intermediate commodities price vectors for vertically hyper-integrated sector $i$ is given by:

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}_k^{(i)T} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{3.5}$$

and therefore

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}_k^{(i)T}(\mathbf{I}-\mathbf{M}(\pi-c_i))^{-1} = \overline{w}\mathbf{z}_k^{(i)T}\mathbf{\Phi}^{(i)}(\pi) \tag{3.6}$$

Full employment of the labour force and full expenditure of income — the *flows* of the economic system — are guaranteed by a *macroeconomic condition*, analytically emerging as a condition for systems (3.1) and (3.2) to have non trivial solutions. Such a condition, though derived within a multi-sectoral framework, is independent of the number of sectors conforming the economic system as a whole, and therefore emerges as being a truly *macroeconomic* condition. Pasinetti (1981) calls it *effective demand condition*:

$$\mathbf{a}_{ni,t}^T\mathbf{a}_{in} + \mathbf{a}_{ni}^T\overline{\mathbf{M}}\mathbf{a}_{in} + \mathbf{a}_{ni}^T\overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in} \equiv \overline{\mathbf{z}}^T\mathbf{a}_{in} = 1 \tag{3.7}$$

[2]For details, see Garbellini (2010b).

On the contrary, full utilisation of the productive capacity — the *stocks* of the economic system — is guaranteed by a whole *series of sectoral conditions*, ensuring that the number of units of productive capacity available *at the beginning* of the time period is exactly that necessary for the satisfaction of final demand:

$$\mathbf{x} = \mathbf{k} \tag{3.8}$$

Moreover, it is worth recalling the definitions of vertically hyper-integrated productive capacity and labour.

A unit of vertically hyper-integrated productive capacity for sector $i$ is the set of all intermediate commodities directly, indirectly, and *hyper-indirectly* needed for the production of one unit of commodity $i$ as a final consumption good.

In the same way, the vertically hyper-integrated labour coefficient for sector $i$ is the amount of labour directly, indirectly, and *hyper-indirectly* needed for the production of one unit of commodity $i$ as a consumption good.

We can now go on and set up the general multi-sector dynamic model.

# 4 Setting up a general multi-sector dynamic model

Now, following Pasinetti (1981, section 1, chapter V) we shall define the *initial conditions* of the economic system and the *laws of motion* of the main economic variables.

At time 0, the economic system is characterised by:

(i) A series of $m$ *stocks* of intermediate commodities expressed in units of vertically hyper-integrated productive capacity:

$$k_{i,0}, \qquad (i = 1, 2, \ldots, m); \tag{4.1}$$

(ii) an *exogenous* population $\overline{x}_{n,0}$;

(iii) a series of $m$ technical coefficients, representing the quantity of labour *directly* necessary for the production of one unit of each of the $m$ commodities produced in the economic system as a whole:

$$a_{ni,0}, \qquad (i = 1, 2, \ldots, m); \tag{4.2}$$

(iv) a series of $m$ *per capita* (average) consumption coefficients:

$$a_{in,0}, \qquad (i = 1, 2, \ldots, m); \tag{4.3}$$

(v) a series of $m$ investment coefficients, i.e. the average per capita demand for the $m$ commodities produced in the economic system as new investment goods:

$$a_{k_i n,0}, \qquad (i = 1, 2, \ldots, m); \tag{4.4}$$

At time zero, all these coefficients are such as to satisfy the relations defining equilibrium in the economic system as a whole. This means that all the technical, consumption and investment coefficients under (iii), (iv) and (v) are such as to satisfy *macroeconomic condition* (3.7) at time zero:[3]

$$\mathbf{a}_{ni,t}^{T}\mathbf{a}_{in,t} + \mathbf{a}_{ni,t}^{T}\overline{\mathbf{M}}\mathbf{a}_{in,t} + \mathbf{a}_{ni,t}^{T}\overline{\mathbf{M}}\widehat{\mathbf{c}}_{t+1}\mathbf{a}_{in,t} = 1, \qquad t = 0 \tag{4.6}$$

Moreover, this means that — in each vertically hyper-integrated sector $i$, ($i = 1, 2, \ldots, m$) — the stocks of capital goods expressed in units of vertical hyper-integrated productive capacity under (i) satisfy the series of sectoral conditions (3.8) for full utilisation of productive capacity, i.e.:

$$k_{i,0} = x_{i,0} \qquad \forall i = 1, 2, \ldots, m \tag{4.7}$$

*Macroeconomic* condition (4.6), referring to the *flows* of the economic system, and the series of *sectoral* conditions (4.7), referring to the *stocks*, are the equilibrium condition within a single period of time. Of course, the fact that they might be satisfied within a single time period $t$ does not imply that any automatism will enable the economic system to do the same in the following time periods as well.

In order to define the concept of equilibrium in a dynamic framework,[4] we now have to describe the way in which the relevant variables move through time.

Differently from what Pasinetti (1981) did, we will describe such movements using *discrete*, rather than continuous, time. This will introduce some additional

[3]Notice that, in principle, matrix $\overline{\mathbf{M}}$ should be dated too — even if we are making the assumption that inter-industry coefficients in matrix $\mathbf{A}$ are not changing — since it depends on the rates of growth of sectoral per capita demand, which — as we are going to see in a moment — are themselves changing through time.

Anyway, from one period to the following one, we will consider matrix $\overline{\mathbf{M}}$ as constant too, since the change is very small. Specifically:

$$\mathbf{M}_{t}^{(i)} - \mathbf{M}_{t-1}^{(i)} = \mathbf{H}^{2} r_{i,t}\sigma_{r_{i,t+1}} \Big(\mathbf{I} + \mathbf{H} r_{i,t}(2 + \sigma_{r_{i,t+1}}) + \mathbf{H}^{2} r_{i,t+1}^{2}(3 + 3\sigma_{r_{i,t+1}} + \sigma_{r_{i,t+1}}^{2}) + \ldots\Big)$$

where

$$\sigma_{r_{i,t}} = (r_{i,t} - r_{i,t-1})/r_{i,t-1} \tag{4.5}$$

i.e. $\sigma_{r_{i,t}}$ is the *speed* with which per capita demand for consumption commodity $i$ ($i = 1, 2, \ldots, m$) changes through time.

The order of magnitude of this difference is clearly very small, though not necessarily irrelevant. For the time being, however, in order not to complicate too much notation and derivation, we will assume the $\mathbf{M}^{(i)}$, and thus also $\overline{\mathbf{M}}$, to be constant through time.

[4]As already explained elsewhere (see Garbellini & Wirkierman 2010b), here we do not have an equilibrium position which is automatically maintained through time; rather, we have *a series* of *situations* of equilibrium, which have to be actively pursued through the choice of an appropriate amount of new investments.

analytical complications, but — together with the re-introduction of the whole set of inter-industry relations — will also allow us to analyse more in detail the dynamics of the main economic variables.

The dynamics of population and of labour, demand, and investment coefficients are the following:

(i) Population increases over time at a *steady* rate $g$:

$$\overline{x}_{n,t} = \overline{x}_{n,0}(1+g)^t \tag{4.8}$$

(ii) Direct labour coefficients change through time at the *non-steady* rates $\varrho_{i,t}$, which are different from sector to sector:

$$a_{ni,t} = a_{ni,t-1}(1-\varrho_{i,t}), \qquad i = 1, 2, \ldots, m \tag{4.9}$$

(iii) Demand coefficients change through time at the *non-steady* rates $r_{i,t}$, different from sector to sector:

$$a_{in,t} = a_{in,t-1}(1+r_{i,t}) \tag{4.10}$$

Of course, demand and labour coefficients cannot be negative, since this would have no economic meaning.

As stated above, no automatism guarantees that, once satisfied at time zero, conditions (3.7) and (3.8) continue to hold also for $t = 1, 2, \ldots$. The dynamics under (i), (ii) and (iii) are such as to continually change the structure of the net output and of relative labour productivities; therefore, full employment of the labour force and of productive capacity, together with full expenditure of income, are tasks to be actively pursued through *institutional mechanisms*.

What this framework can tell us is which conditions, *if satisfied* — and *given* each time period's coefficients — allow us to move the economic system from the equilibrium position entailed by the structure of the economic system in one time period to that entailed by the structure of the following one.

By dating all magnitudes whose movements through time have just been introduced, effective demand condition 3.7 can be written as:

$$\mathbf{a}_{ni,t}^T \mathbf{a}_{in,t} + \mathbf{a}_{ni,t}^T \overline{\mathbf{M}} \mathbf{a}_{in,t} + \mathbf{a}_{ni,t}^T \overline{\mathbf{M}} \widehat{\mathbf{c}}_{t+1} \mathbf{a}_{in,t} = 1 \tag{4.11}$$

At this stage of the analysis, Pasinetti (1981) derived what he then called the *capital accumulation conditions*, a series of *sectoral* conditions concerning equilibrium new investments, i.e. guaranteeing the evolution through time of the number of units of vertically hyper-integrated productive capacity available at the beginning of each time period in line with the evolution of final demand (for consumption commodities):

$$a_{k_i n,t} = c_{i,t+1} a_{in,t}, \qquad i = 1, 2, \ldots, m$$

Since we have started developing the reformulation of the whole framework from the more general analytical formulation presented in Pasinetti (1988), where such conditions — and their derivation — were already taken for granted, they have been already introduced in the price and quantity systems, and therefore in the macroeconomic condition (4.11). Anyway, it is worth doing a step backwards in order to explicitly derive them as conditions for 'equilibrium' capital accumulation.

As we hinted at before, these conditions, *if* satisfied, allow to keep productive capacity fully utilised period after period, i.e. drives *capital accumulation* in line with the evolution of final demand for consumption commodities — and with *technical progress*.

The variation of the stock of capital available in each vertically hyper-integrated sector $i$ at the beginning time period $t$ is given by the amount of intermediate commodities bought for the sake of new investment in the previous one. If stock equilibrium is to be maintained from time period $t$ to $t+1$, the number of units of vertically hyper-integrated productive capacity to be devoted to the expansion of productive capacity itself must be the same as the variation of demand for the corresponding consumption good. I.e.:

$$k_{i,t+1} - k_{i,t} = x_{i,t+1} - x_{i,t} = a_{k_in,t}\overline{x}_{n,t} \tag{4.12}$$

and hence:

$$a_{in,t+1}\overline{x}_{n,t+1} - a_{in,t}\overline{x}_{n,t} = a_{k_in,t}\overline{x}_{n,t} \tag{4.13}$$

By using expressions (4.8) and (4.10) and rearranging, what we get is:

$$a_{k_in,t} = (g + r_{i,t+1})a_{in,t} = c_{i,t+1}a_{in,t} \tag{4.14}$$

i.e. Pasinetti's (1981) capital accumulation conditions, though formulated in *discrete time*.

Equilibrium investments in period $t$ are therefore determined by the expansion of demand from $t$ to $t+1$, and are those investments which ensure the expansion of productive capacity to be exactly in line with the movements of sectoral total demand for consumption commodities.

Expression (4.14) shows the advantage of using discrete, rather than continuous, time. The capital accumulation conditions originally derived by Pasinetti (1981, Chapter V, p. 86) are slightly different from (4.14), i.e.:

$$a_{k_in}(t) = (g + r_i)a_{in}(t)$$

Using continuous rather than discrete time is a matter of analytical simplicity. When such a choice is made, assuming non-steady rates of change of variables does not make sense, since it would introduce exactly the same type of analytical complications that carrying out a continuous time analysis is intended to avoid.[5] As a consequence, steady rates of growth are thus assumed, making impossible to distinguish between current, past and future rates of change of per capita demand. On the contrary, by using discrete time and hence allowing for non-steady rates of change, it is possible to make it clear that new investment do not depend on the change of demand from period $t-1$ to $t$, but from $t$ to $t+1$, i.e. on the change of demand from the time period in which the investment decision has to be taken, to the following one.

# 5 Direct, indirect and hyper-indirect labour productivity

Item (ii) in the list of the dynamics depicted in the previous section concerns the movements of *direct labour* requirements. However, in the present framework, we are not only interested in these coefficient, but also in the vertically hyper-integrated ones. The latter can be written as:[6]

$$z_{i,t}^{*} = a_{ni,t} + \mathbf{a}_{ni,t}^{T}\mathbf{m}_{i}^{*} + c_{i,t+1}\mathbf{a}_{ni,t}^{T}\mathbf{m}_{i}^{*}, \qquad i = 1, 2, \ldots, m \tag{5.1}$$

or

$$\overline{\mathbf{z}}^{T} = \mathbf{a}_{ni,t}^{T} + \mathbf{a}_{ni,t}^{T}\overline{\mathbf{M}} + \mathbf{a}_{ni,t}^{T}\overline{\mathbf{M}}_{t}\widehat{\mathbf{c}}_{t+1} \tag{5.2}$$

For each vertically hyper-integrated sector $i$ ($i = 1, 2, \ldots, m$) expression (4.9) describes the movement of the *first* addendum of this sum, whose rate of change from time period $t-1$ to $t$ is given by $\varrho_{i,t}$. The second addendum is *indirect labour*, i.e. labour indirectly required to replace *all* those intermediate commodities used up during the production process for producing *both* final consumption commodity $i$ *and* the whole *set* of intermediate commodities to be devoted to new investment. The third one is *hyper-indirect labour*, i.e. the quantity of labour necessary for the production of additional productive capacity. By defining $\varrho_{k_i,t}$ and $\varrho_{k_{hi},t}$, respectively, the rates of change of the second and third addenda, the rate of change of the whole vertically hyper-integrated labour coefficient can be written as:

$$\varrho_{z_i,t}' = \varrho_{i,t}\frac{a_{ni,t-1}}{z_{i,t-1}} + \varrho_{k_i,t}\frac{\mathbf{a}_{ni,t-1}^{T}\mathbf{m}_{i}^{*}}{z_{i,t-1}} + \varrho_{k_i^{c_i},t}\frac{c_{i,t+1}\mathbf{a}_{ni,t-1}^{T}\mathbf{m}_{i}^{*}}{z_{i,t-1}} \tag{5.3}$$

[5] For a discussion on this point, see Garbellini (2010c, section 3.4).

[6] To see how this decomposition can be derived, see Garbellini (2010b, section 4).

i.e. as the *weighted average* of the rates of change of the three addenda, the weights being the proportion of each of them to the total.

Clearly, both $\varrho_{k_i,t}$ and $\varrho_{k_{hi},t}$ are themselves weighted averages of the direct labour coefficients. More specifically, the rate of change of indirect labour productivity is given by:

$$\varrho_{k_i,t} = \frac{\mathbf{a}^T_{ni,t-1}\widehat{\boldsymbol{\varrho}}_t\mathbf{m}^*_i}{\mathbf{a}^T_{ni,t-1}\mathbf{m}^*_i}, \qquad i = 1, 2, \ldots, m \tag{5.4}$$

i.e. by the weighted average of the $\varrho_i$'s, the weights being the ratios, at time $t-1$, of the direct labour necessary to produce each commodity $i$ ($i = 1, 2, \ldots, m$) entering $\mathbf{m}^*_i$ to the *total* direct labour necessary to produce the whole unit of vertically hyper-integrated productive capacity.

Finally, the rate of change of hyper-indirect labour productivity is given by:

$$\varrho_{k_i^{c_i},t} = \frac{\mathbf{a}^T_{ni,t}\mathbf{m}^*_i c_{i,t+1} - \mathbf{a}^T_{n,t-1}\mathbf{m}^* c_{i,t}}{\mathbf{a}^T_{n,t-1}\mathbf{m}^*_i c_{i,t}} = \frac{\mathbf{a}^T_{ni,t}(\widehat{\boldsymbol{\varrho}} - \sigma_{r_i,t}\mathbf{I})\mathbf{m}^*_i}{\mathbf{a}^T_{ni,t}\mathbf{m}^*_i} \tag{5.5}$$

where $\sigma_{r_i,t}$ is given by expression (4.5). Rearranging and substituting expression (5.4) into equation (5.5), the latter can be rewritten as:

$$\varrho_{k_i^{c_i},t} = \varrho_{k_i,t} - \sigma_{r_i,t+1} \tag{5.6}$$

i.e., hyper-indirect labour increases or decreases with respect to indirect one in proportion to the *speed* of change of per-capita demand for the corresponding final consumption commodity. This means that overall labour productivity — intended as the amount of working hours necessary to produce one unit of the corresponding consumption commodity and to make it possible to keep demand satisfied in the following period too — increases or decreases, *ceteris paribus*, when the growth of demand *decelerates* or *accelerates*, respectively.

This opens up the question of whether or not the vertically hyper-integrated labour coefficients are a good measure for labour productivity, since they are influenced not only by technical coefficients, but also by the movements through time of demand for consumption goods. The order of magnitude of this last component is likely to be very small, but when dealing with sectors experiencing great expansion (or contraction) this might not necessarily be so.[7]

Further decompositions of the vertically hyper-indirect labour coefficients can be obtained, when useful for specific tasks. For example, we could write $z_{i,t}$ as:

$$z_{i,t} = a_{ni,t} + \mathbf{a}^T_{ni,t}\mathbf{h}_i + \mathbf{a}^T_{ni,t}c_{i,t+1}\mathbf{m}^*_i + \mathbf{a}^T_{ni,t}\mathbf{H}c_{i,t+1}\mathbf{m}^*_i \tag{5.7}$$

[7]The theoretical and empirical problems connected to the measurement of productivity changes in vertically integrated and vertically *hyper*-integrated terms are treated more in depth in Garbellini & Wirkierman (2010a).

where the first and second addenda are, respectively, direct and indirect labour for the production of consumption commodity $i$ — i.e. vertically integrated labour — while the third and fourth are direct and indirect labour for the production of additional productive capacity — i.e. vertically integrated labour for the production of additional productive capacity:

$$z_{i,t} = \mathbf{a}_{ni,t}^T(\mathbf{I}+\mathbf{H})\mathbf{e}^{(i)} + \mathbf{a}_{ni,t}^T(\mathbf{I}+\mathbf{H})c_{i,t+1}\overline{\mathbf{M}}\mathbf{e}^{(i)} \tag{5.8}$$

This last decomposition gives us a further hint about the differences between the vertically integrated and the vertically hyper-integrated approach. In the former, indirect labour coefficients simply indicate that amount of working time devoted, both directly and indirectly, to replace the intermediate commodities used up for the production of one unit of the final consumption commodity produced in the (vertically integrated) sector: $\mathbf{a}_{ni.t}^T\mathbf{H}$. On the contrary, what Pasinetti (1988) calls indirect labour is something more: it is the amount of working hours directly and indirectly necessary for the replacement of intermediate commodities used up for the production of one unit of final consumption commodity $i$ ($i = 1, 2, \ldots, m$) — $\mathbf{a}_{ni,t}^T\mathbf{H}\mathbf{e}^{(i)}$ — *and* for the production of the whole set of intermediate commodities composing one unit of the corresponding (additional) vertically hyper-integrated productive capacity — $\mathbf{a}_{ni,t}^T\mathbf{H}\mathbf{m}_i^*$.

Moreover, Pasinetti's (1973, section 9) paper explicitly treats higher order vertical integration, where by vertically integrated sectors of second order are the vertically integrated sectors producing the units of productive capacity. Expression (5.8) therefore shows that vertically hyper-integrated labour is the sum of vertically integrated labour of first and second order, when by productive capacity we mean vertically hyper-integrated one.

Expression (5.8) can also help us in dealing with the problem of labour productivity: each vertically hyper-integrated labour coefficient is decomposed in two parts. While the second is influenced by the movements of demand, the first one reflects purely technological factors. The relationships between the two gives us an idea of the weight of new investments, i.e. of *capital accumulation*, on the production effort to be put forward by each vertically hyper-integrated sector to keep equilibrium through time.

Finally, we can compute the rate of change of the *labour equivalents*[8] — $\varrho_{i,t}^{(e)}$ and $\varrho_{k_i,t}^{(e)}$ ($i = 1, 2, \ldots, m$), for consumption commodities and units of productive capacity, respectively — such that

$$z_{i,t}^{(e)} = (1 - \varrho_{i,t}^{(e)})z_{i,t-1}^{(e)} \tag{5.9}$$

[8]For a definition of labour equivalents and of labour transformation matrix see Garbellini (2010b, section 6).

and

$$z^{(e)}_{k_i,t} = (1 - \varrho^{(e)}_{k_i,t}) z^{(e)}_{k_i,t-1}, \qquad i = 1, 2, \dots, m \tag{5.10}$$

By recalling that $z^{(e)}_{i,t} = \overline{\mathbf{z}}^T_t \boldsymbol{\phi}_{i,t}(\pi)$ and that $z^{(e)}_{k_i,t} = z^{(e)}_{i,t}\overline{\mathbf{M}} = \overline{\mathbf{z}}^T_t \overline{\mathbf{M}} \boldsymbol{\phi}_{k_i,t}(\pi)$, the expressions for $\varrho^{(e)}_{i,t}$ and $\varrho^{(e)}_{k_i,t}$ can be written as:

$$\varrho^{(e)}_{i,t} = \frac{\overline{\mathbf{z}}^T_{t-1}(\widehat{\boldsymbol{\varrho}}'_t - \widehat{\boldsymbol{\sigma}}_{\phi_{i,t}})\boldsymbol{\phi}_{i,t-1}(\pi)}{\overline{\mathbf{z}}^T_{t-1}\boldsymbol{\phi}_{i,t-1}(\pi)} = \frac{\overline{\mathbf{z}}^T_{t-1}(\widehat{\boldsymbol{\varrho}}'_t - \widehat{\boldsymbol{\sigma}}_{\phi_{i,t}})\boldsymbol{\phi}_{i,t-1}(\pi)}{z^{(e)}_{i,t}} \tag{5.11}$$

and

$$\varrho^{(e)}_{k_i,t} = \frac{\overline{\mathbf{z}}^T_{t-1}(\widehat{\boldsymbol{\varrho}}' - \widehat{\boldsymbol{\sigma}}_{\phi_{i,t}})\boldsymbol{\phi}_{i,t-1}(\pi)\overline{\mathbf{M}}}{\overline{\mathbf{z}}^T_{t-1}\boldsymbol{\phi}_{i,t-1}(\pi)\overline{\mathbf{M}}} = \frac{\overline{\mathbf{z}}^T_{t-1}(\widehat{\boldsymbol{\varrho}}' - \widehat{\boldsymbol{\sigma}}_{\phi_{i,t}})\overline{\mathbf{M}}\boldsymbol{\phi}_{k_i,t-1}(\pi)}{z^{(e)}_{k_i,t}} \tag{5.12}$$

where $\widehat{\boldsymbol{\sigma}}_{\phi_{i,t}}$ is a diagonal matrices whose elements are the rates of change from time period $t-1$ to $t$ of the corresponding elements of vector $\boldsymbol{\phi}_{i,t}(\pi)$.

Both $\varrho^{(e)}_{i,t}$ and $\varrho^{(e)}_{k_i,t}$ are weighted averages of the difference between the rate of change of each element of column $i$ of the labour transformation matrix and the corresponding rate of growth of vertically hyper-integrated labour.

Finally, it is worth specifying a series of 'hypothetical' magnitudes, to use Pasinetti's (1988) terminology, which are associated to those elements of vectors $\mathbf{z}^{(i)T}$ $(i = 1, 2, \dots, m)$ different from the $i$-th one.

We have defined $\varrho'_i$ as the rate of change of labour productivity in vertically hyper-integrated sector $i$. I.e., $\varrho'_i$ is the opposite of the rate of change of the corresponding vertically hyper-integrated labour coefficient $z^*_i$, which is the $i$-th element of vector $\mathbf{z}^{(i)T}$; we should also define the rate of change of the remaining $m-1$ elements. Let therefore $-\varrho^{(i)\prime}_j$ be the rate of change through time of the $j$-th element of vector $\mathbf{z}^{(i)T}$, with $i = 1, 2, \dots, m$ and $j \neq i$.

# 6 The price system: choice of the *numéraire* and degrees of freedom

Before analysing the dynamics of relative physical quantities and relative prices, it is worth spending a few words on the price system, and on the meaning, when time is inserted into the picture, of choosing a *numéraire*.

As explained by Pasinetti (1981, Chapter V, section 12), the price system is characterised by *two* degrees of freedom, one concerning the *initial price* of the *numéraire* commodity, and one concerning the *rate of change* of such a price through time.

In particular, choosing labour as the *numéraire* commodity — and therefore keeping the wage rate fixed — actually means closing these two degrees of freedom as follows:

$$\begin{cases} w_0 = \overline{w} = 1 \\ \sigma_{w,t} = 0, \quad \forall t = 1, 2, \dots \end{cases} \tag{6.1}$$

But we can also choose the price of any commodity, or composite commodity, as the *numéraire* of the price system. If, for example, we chose commodity $h$ as a basis, this would amount at setting:

$$\begin{cases} p_{h,0} = 1 \\ \sigma^{(h)}_{p_{h,t}} = 0, \quad \forall t = 1, 2, \dots \end{cases} \tag{6.2}$$

In order to express all prices in terms of commodity $h$, what we are left to do is expressing both the (real) wage rate at time zero and its rate of change through time *in terms of commodity* $h$ itself, i.e.:

$$\begin{cases} w_0^{(h)} = \dfrac{1}{\overline{\mathbf{z}}_0^T \left(\mathbf{I} - \overline{\mathbf{M}}(\pi \mathbf{I} - \widehat{\mathbf{c}}_1)\right)^{-1} \mathbf{e}^{(h)}} = \dfrac{1}{z^e_{h,0}(\pi)} \\ \\ \sigma^{(h)}_{w,t} = -\varrho^{(e)\prime}_{h,t} \end{cases} \tag{6.3}$$

The real wage rate increases/decreases in the same proportion as the labour *equivalent* content of the *numéraire* commodity decreases/increases.

By inserting expression (6.3) into the price system, all prices will automatically be expressed in terms of commodity $h$; in the same way, by inserting expression (6.1), all prices will automatically be expressed in terms of labour.

# 7 Structural dynamics of physical quantities and commodity prices

We can now explicitly state the dynamics of both *relative* physical quantities and prices, respectively:

$$\begin{cases} \mathbf{x}_t = \mathbf{a}_{in,t}\overline{x}_{n,t} = (\mathbf{I} + \widehat{\mathbf{c}}_t)\mathbf{a}_{in,t-1}\overline{x}_{n,t-1} \\ \mathbf{x}_{k,t} = (\mathbf{I} + \widehat{\mathbf{c}}_{t+1})\mathbf{a}_{in,t}\overline{x}_{n,t} = (\mathbf{I} + \widehat{\mathbf{c}}_t + \widehat{\mathbf{c}}_{t+1})\mathbf{a}_{in,t-1}\overline{x}_{n,t-1} \end{cases} \tag{7.1}$$

$$\begin{cases} \mathbf{p}_t^T = \overline{w}\mathbf{z}_t^{eT} = \overline{w}\mathbf{z}_{t-1}^{eT}(\mathbf{I} - \widehat{\boldsymbol{\varrho}}^{(e)}_{i,t}) \\ \mathbf{p}_{k,t}^T = \overline{w}\mathbf{z}_{k,t}^{eT} = \overline{w}\mathbf{z}_{k,t-1}^{eT}(\mathbf{I} - \widehat{\boldsymbol{\varrho}}^{(e)}_{k_i,t})\varrho^{(e)}_{k_i,t} \end{cases} \tag{7.2}$$

the corresponding rates of change through time therefore being:

$$\begin{cases} \sigma_{x_i,t} = c_{i,t} \\ \sigma_{x_{k_i},t} = c_{i,t} + c_{i,t+1} \end{cases} \qquad i = 1, 2, \ldots, m \tag{7.3}$$

$$\begin{cases} \sigma_{p_i,t} = -\varrho^{(e)}_{i,t} \\ \sigma_{p_{k_i},t} = -\varrho^{(e)}_{k_i,t} \end{cases} \qquad i = 1, 2, \ldots, m \tag{7.4}$$

As it is apparent from expressions (7.1) and (7.2), the whole structure of physical quantities and relative prices continuously changes through time, since all the rates of change — of per capita demand and of labour productivity — are different from sector to sector, and from time period to time period.

The very fact of having introduced the more complete description of technology, i.e. the complete matrix of inter-industry relations, and of having chosen to use discrete, rather than continuous, time, make such dynamics much more complex than in Pasinetti's (1981, Chapter V, p. 92) original formulation.

As to the quantity system, the analytical formulation of the dynamics of physical quantities is not affected by the introduction of the complete matrix $\mathbf{A}$: we are working with units of vertically hyper-integrated productive capacity, which still are such whatever their physical content. Clearly, changes in the technique affect this physical content. But this problem, thanks to the adoption of these particular units of measurement for intermediate commodities, can be kept separate from that of capital accumulation. As Pasinetti himself states:

> [T]he notion of a physical unit of productive capacity, by being defined with reference to the commodity that is produced, continues to make sense, as a physical unit, whatever complications technical change may cause to its composition in terms of ordinary commodities.
>
> (Pasinetti 1973, p. 24).

On the contrary, the fact of introducing discrete, rather than continuous, time — and therefore of having the possibility of taking into account changes, from time period to time period, of the various rates of growth — makes it clear that the rate of change of the *number* of units of productive capacity to be produced in each vertically hyper-integrated sector in time period $t$ crucially depends on the rate of change of demand for the corresponding consumption commodity *both* from time period $t-1$ to $t$ *and* from time period $t$ to $t+1$.

As to the price system, things are much more complicated than for physical quantities, since, *with the complete description the technique in use*, the price of each consumption commodity, and of the corresponding unit of vertically hyper-integrated productive capacity, comes to depend on the prices of *all* the others, which are directly or indirectly used in the corresponding vertically hyper-integrated sector. The dynamics of prices, as clearly emerges from expression

(5.4), depend on the change in productivity in *all* industries entering the sector, as well as on the difference between the *future* rate of change of demand for consumption commodities — that is to say, the rate of new investment required for the system to keep productive capacity fully utilised, and therefore to be in stock equilibrium, period after period — and the rate of profit, which determines the amount of resources that are available, at the end of each time period, for new investments themselves.

In particular, changes in the $r_i$'s, $(i = 1, 2, \ldots, m)$ affect both the vertically hyper-integrated labour coefficients — which increase as future demand for the corresponding consumption commodity increases with respect to the current one — and matrices $\mathbf{\Phi}(\pi)$ and $\mathbf{\Phi}_k(\pi)$.

The change through time of the elements of matrix $\mathbf{\Phi}(\pi)$ (and $\mathbf{\Phi}_k(\pi)$) itself are analytically quite complex, and therefore cannot be explicitly computed and singled out. Anyway, such elements do not change directly together with the rates of change of per capita demand, but rather through the difference between the rate(s) of profit and the rate of change of total demand for each consumption commodity, i.e. $(g + r_{i,t+1})$, $(i = 1, 2, \ldots, m)$.

Hence, the price of any consumption commodity — and of the corresponding units of vertically hyper-integrated productive capacity — depend on the rates of change of demand for the corresponding consumption commodity and of labour productivity in *all* sectors. The value, at current prices, created in the economic system as a whole as a consequence of production activity is distributed among the different sectors according to whether their own rate of growth, *and* those of the others, are greater than, smaller than, or equal to the rate of profit.[9]

## 8 Sectoral and aggregate magnitudes through time

After discussing the determinants of the movements through time of relative quantities and prices, let us now analyse the structural dynamics of some other relevant economic magnitudes, namely the *capital/output* ratio(s), the *capital/labour* ratio(s), and the product per worker.

As stated elsewhere,[10] there is a deep difference between the concept of capital intensity, as summarised by sectoral capital/output ratios, and degree of mechanisation, as expressed by sectoral capital/labour ratios.

The capital/output ratio for vertically hyper-integrated sector $i$ is the ratio of

[9] For details on the shifts of (vertically hyper-integrated) labour value from sector to sector as a consequence of the rate of profit being different from (sectoral) rates of change o demand for consumption commodities, see Garbellini (2010b, section 6).

[10] See Pasinetti (1981, Chapter IX, sections 4-7) and Garbellini (2010b, section 7).

two quantities of *labour equivalents*: the labour equivalent associated to the production of the units of vertically hyper-integrated productive capacity available at the beginning of the time period — i.e. $k_{i,t}$ units of productive capacity, evaluated at their current price $\overline{w}z^e_{k_i,t}(\pi)$ — and the labour equivalent associated to the production of the final (consumption) commodity — i.e. $x_{i,t}$ units od commodity $i$ evaluated at its current price $\overline{w}z^e_{i,t}(\pi)$.

The wage rate — appearing both at the numerator and at the denominator — cancels out, and therefore its dynamics does not affect the movements of the capital/output ratios. Anyway the rate of profit, or better, the difference between the rate of profit and the rate of growth of each vertically hyper-integrated sector, does. In fact, matrices $\mathbf{\Phi}_t(\pi)$ and $\mathbf{\Phi}_{k,t}(\pi)$ depend on the actual rate of profit, and therefore labour equivalents $z^e_{i,t}(\pi)$ and $z^e_{k_i,t}(\pi)$ do depend on it as well.

The information provided by the degree of capital intensity, as Pasinetti (1981) explains in detail, are relevant for two kinds of problems. The first concerns the effect of sectoral investments on the flow of net production. The second concerns the process of price formation, since the higher the capital/output ratio in sector $i$, the higher the incidence of capital — i.e. of profit mark-up on the stock of existing capital — in the price of the corresponding final commodity.

In particular, the sectoral capital/output ratio for vertically hyper-integrated sector $i$ $(i = 1, 2, \ldots, m)$ can be written as:[11]

$$\gamma_{i,t} = \frac{z^e_{k_i,t}(\pi)k_{i,t}}{z^e_{i,t}(\pi)x_{i,t}} = \frac{z^e_{k_i,t}(\pi)}{z^e_{i,t}(\pi)} = \frac{z^e_{k_i,t-1}(\pi)(1-\varrho^{(e)}_{k_i,t})}{z^e_{i,t-1}(\pi)(1-\varrho^{(e)}_{i,t})} \tag{8.1}$$

its rate of change through time therefore being:

$$\sigma_{\gamma_i,t} = \frac{\varrho^{(e)}_{i,t} - \varrho^{(e)}_{k_i,t}}{1-\varrho^{(e)}_{i,t}} \tag{8.2}$$

By looking at expression (8.2), we see that the dynamics of the degree of capital intensity in each vertically hyper-integrated sector $i$ $(i = 1, 2, \ldots, m)$ crucially depend on the rate of change of the labour equivalent for both the final consumption commodity and the corresponding units of vertically hyper-integrated productive

[11] The second equality in (8.1) reflects the fact that, since we are looking for those conditions that allow to keep the economic system in equilibrium, we start from a situation of stock-equilibrium, in which $k_{i,t} = x_{i,t}$. Anyway, if we wanted to analyse a specific, concrete situation, using actual data, it would be quite likely that $k_{i,t} \neq x_{i,t}$. See Garbellini & Wirkierman (2010b) for details.

capacity. In particular, $\sigma_{\gamma_i,t} \lesseqgtr 0$ according to whether

$$\varrho^{(e)}_{i,t} \lesseqgtr \frac{1+\varrho^{(e)}_{k_i,t}}{2}$$

Recalling expressions (5.11) and (5.12), we see that the rates of change through time of the labour equivalents $\varrho^{(e)}_{i,t}$ and $\varrho^{(e)}_{k_i,t}$ depend not only on labour productivity in the corresponding vertically hyper-integrated sector, but also on the changes in the elements of the corresponding columns of matrices $\mathbf{\Phi}(\pi)$ and $\mathbf{\Phi}_k(\pi)$, i.e. on how demand for consumption commodity $i$ changes, from $t$ to $t+1$, with respect to the rate of profit. This in its turn depends not only on whether demand decreases or increases, but also on the speed with which such a decrease or increase takes place.

At the aggregate level, the capital/output ratio can be written as:

$$\Gamma_t = \frac{\mathbf{z}^{eT}_{k_i,t}\mathbf{a}_{in,t}}{\mathbf{z}^{eT}_{i,t}\mathbf{a}_{in,t}} \simeq \frac{\mathbf{z}^{eT}_{k_i,t-1}\left(\mathbf{I}-\widehat{\boldsymbol{\varrho}}^{(e)}_{k_i,t}+\widehat{\mathbf{r}}_{i,t}\right)\mathbf{a}_{in,t-1}}{\mathbf{z}^{eT}_{i,t-1}\left(\mathbf{I}-\widehat{\boldsymbol{\varrho}}^{(e)}_{i,t}+\widehat{\mathbf{r}}_{i,t}\right)\mathbf{a}_{in,t-1}} \tag{8.3}$$

its rate of change through time therefore being a quite complicated expression:

$$\sigma_{\Gamma_t} = \frac{\mathbf{z}^{eT}_{k_i,t-1}\left((\mathbf{I}-\widehat{\boldsymbol{\varrho}}^{(e)}_{k_i,t}+\widehat{\mathbf{r}}_{i,t})\mathbf{a}_{in,t-1}\mathbf{z}^{eT}_{t-1}-\mathbf{a}_{in,t-1}\mathbf{z}^{eT}_{t-1}(\mathbf{I}-\widehat{\boldsymbol{\varrho}}^{(e)}_{i,t}+\widehat{\mathbf{r}}_{i,t})\right)\mathbf{a}_{in,t-1}}{\mathbf{z}^{eT}_{t-1}(\mathbf{I}-\widehat{\boldsymbol{\varrho}}^{(e)}_{i,t}+\widehat{\mathbf{r}}_{i,t})\mathbf{a}_{in,t-1}\mathbf{z}^{eT}_{k_i,t-1}\mathbf{a}_{in,t-1}} \tag{8.4}$$

Clearly, the structural dynamics of the aggregate ratio is much more complicated than that of the sectoral ones, since it depends not only on technology — and income distribution — but also on the whole *structure* of final demand for consumption commodities. Actually, starting from a multi-sectoral framework, it is possible to see how macroeconomic magnitudes and their dynamics crucially depend on an extraordinarily complicated interaction at the level of the sectoral ones; their determination depending not only on technology and income distribution, but also on the very sectoral structure of the economic system as a whole.

Moreover, expression (8.4) shows that the aggregate capital intensity depends not only on the determinants of changes in the labour equivalent associated to the production of *all* consumption commodities and units of vertically hyper-integrated productive capacity, but also on the rate of change of demand for the final consumption commodities produced by *all* sectors.

As to the capital/labour ratios, the information they provide is useful in facing problems concerning labour employment; more precisely, those problems relating technical progress and employment:

> Changes in the degree of mechanisation, as expressed by the capital labour ratio, mean changes in the size of employment associated with any given amount of capital goods, expressed at (average) constant prices.
>
> (Pasinetti 1981, p. 183)

The wage rate, differently from the case of the capital/output ratios, cannot be factored out here, since it only appears at the numerator of the sectoral ratios; therefore its dynamics — in addition to those of vertically hyper-integrated labour productivity and of the labour equivalent for the production of the sector-specific unit of vertically hyper-integrated productive capacity — do affect the variations of the sectoral degrees of mechanisation.

Sector $i$'s degree of mechanisation is not the ratio of two quantities of labour equivalent, but the ratio of a quantity of labour equivalent — for the production of the stock of units of vertically hyper-integrated productive capacity available at the beginning of the time period — and a quantity of labour — the vertically hyper-integrated labour necessary for the production of $x_{i,t}$ units of commodity $i$, the final output of the production process carried out in vertically hyper-integrated sector $i$.

In particular, the capital/labour ratio for vertically hyper-integrated sector $i$ $(i = 1, 2, \ldots, m)$ is given by:

$$\theta_{i,t} = \frac{\overline{w} z^{e}_{k_i,t}(\pi) k_{i,t}}{z^{*}_{i,t} x_{i,t}} = \frac{\overline{w} z^{e}_{k_i,t}(\pi)}{z^{*}_{i,t}} = \frac{\overline{w} z^{e}_{k_i,t-1}(\pi)\left(1 - \varrho^{(e)}_{k_i,t}\right)}{z^{*}_{i,t-1}\left(1 - \varrho'_{i,t}\right)} \tag{8.5}$$

its rates of change through time therefore being:

$$\sigma_{\theta_{i,t}} = \frac{\varrho'_{i,t} - \varrho^{(e)}_{k_i,t}}{1 - \varrho'_{i,t}} \tag{8.6}$$

The capital/labour ratio for sector $i$ $(i = 1, 2, \ldots, m)$ depends negatively on vertically hyper-integrated labour productivity; moreover, it increases together with the corresponding labour equivalent for the production of one unit of productive capacity. As it can be seen, the difference with respect to the rate of change of the capital/output ratios is that the latter involve the rate of change through time of two quantities of labour equivalent — the numerator and the denominator of the ratios themselves — while the former involve the rate of change of a quantity of labour equivalent — for producing the corresponding vertically hyper-integrated productive capacity, i.e. the numerator of the ratios — and of a physical quantity of labour — for producing the final consumption commodity, i.e. the denominator of the ratios.

At the aggregate level, the capital/labour ratio for the economic system as a whole is given by:

$$\Theta_t = \frac{\overline{w}\mathbf{z}_{k,t}^{e^T}\mathbf{a}_{in,t}}{\overline{\mathbf{z}}_t^T\mathbf{a}_{in,t}} = \frac{\overline{w}\mathbf{z}_{k,t-1}^{e^T}\left(\mathbf{I} - \widehat{\boldsymbol{\varrho}}_{k_i,t}^{(e)} + \widehat{\mathbf{r}}_t\right)\mathbf{a}_{in,t-1}}{\overline{\mathbf{z}}_{t-1}^T\left(\mathbf{I} - \widehat{\boldsymbol{\varrho}}_t' + \widehat{\mathbf{r}}_t\right)\mathbf{a}_{in,t-1}} \tag{8.7}$$

its rate of change through time again being quite a complex expression:

$$\sigma_{\Theta_{i,t}} = \frac{\mathbf{z}_{k,t-1}^{e^T}\left((\mathbf{I} - \widehat{\boldsymbol{\varrho}}_{k_i,t}^{(e)} + \widehat{\mathbf{r}}_t)\mathbf{a}_{in,t-1}\overline{\mathbf{z}}_{t-1}^T - \mathbf{a}_{in,t-1}\overline{\mathbf{z}}_{t-1}^T(\mathbf{I} - \widehat{\boldsymbol{\varrho}}_t' + \widehat{\mathbf{r}}_t)\right)\mathbf{a}_{in,t-1}}{\overline{\mathbf{z}}_{t-1}^T(\mathbf{I} - \widehat{\boldsymbol{\varrho}}_t' + \widehat{\mathbf{r}}_t)\mathbf{a}_{in,t-1}\mathbf{z}_{k,t-1}^{e^T}\mathbf{a}_{in,t-1}} \tag{8.8}$$

Also in this case, we can see that the aggregate dynamics is further complicated by the effect of changes in the *composition* of final demand for consumption commodities, interacting with the effect of technological progress and of changes in the rate of profit.

Finally, we can consider the product per worker. The sectoral products are given by:

$$y_{i,t} = \frac{\overline{w}z_{i,t}^e(\pi)}{z_{i,t}^*} = \frac{\overline{w}z_{i,t-1}^e(\pi)(1 - \varrho_{i,t}^{(e)})}{z_{i,t-1}^*(1 - \varrho_{i,t}')} \tag{8.9}$$

their rates of change through time being:

$$\sigma_{y_{i,t}} = \frac{\varrho_{i,t}' - \varrho_{i,t}^{(e)}}{1 - \varrho_{i,t}'} \tag{8.10}$$

Clearly, the product per worker decreases either when there is an increase in the quantity of labour equivalent necessary for the production of consumption commodity $i$ — which, with the corresponding vertically hyper-integrated labour remaining constant, is implied by an increase in the rate of profit with respect to the rate of growth of the sector — or when there is an increase in vertically hyper-integrated labour productivity — which, if the corresponding labour equivalent remains the same, implies exactly the opposite.

The aggregate product per worker is finally given by:

$$Y_t = \frac{\overline{w}\mathbf{z}_t^{e^T}\mathbf{a}_{in,t}}{\overline{\mathbf{z}}_t^T\mathbf{a}_{in,t}} \tag{8.11}$$

its rate of change through time being:

$$\sigma_{Y_{i,t}} = \frac{\mathbf{z}^{eT}_{t-1}\left((\mathbf{I} - \widehat{\boldsymbol{\varrho}}^{(e)}_{i,t} + \widehat{\mathbf{r}}_t)\mathbf{a}_{in,t-1}\mathbf{z}^T_{t-1} - \mathbf{a}_{in,t-1}\overline{\mathbf{z}}^T_{t-1}(\mathbf{I} - \widehat{\boldsymbol{\varrho}}'_t + \widehat{\mathbf{r}}_t)\right)\mathbf{a}_{in,t-1}}{\overline{\mathbf{z}}^T_{t-1}(\mathbf{I} - \widehat{\boldsymbol{\varrho}}^{(e)}_{i,t} + \widehat{\mathbf{r}}_t)\mathbf{a}_{in,t-1}\mathbf{z}^{eT}_{t-1}\mathbf{a}_{in,t-1}} \tag{8.12}$$

To conclude, we might say that there is a deep difference between sectoral and aggregate magnitudes in general. The former only depend on technology and on the specific configuration of income distribution, as well as on the particular movements of the rate of growth of the corresponding vertically hyper-integrated sector with respect to the (uniform or sectoral) rate(s) of profit. The latter crucially depend also on the very structure of the economic system as a whole, and on the way in which such a structure changes through time due to the change in the structure of final demand for consumption goods, and therefore to the related dynamics of capital accumulation — and thus, of technical progress, which clearly affects all the sectoral and aggregate movements taking place in the economic system.

As to aggregate magnitudes, we may add that singling out and isolating the determinants of their movements through time is a very complicated task; a deep understanding of the changes taking place in the economic system can only rely on the joint analysis of the dynamics of sectoral magnitudes. Looking at the aggregates only gives us a very superficial and primitive idea of what is going on at the fundamental level.

# 9 The 'natural' economic system

Expressions (7.1) and (7.2) are such as to satisfy the macroeconomic condition for *flow* equilibrium (4.6), the set of sectoral conditions for *stock* equilibrium (4.7), and *capital accumulation conditions* (4.14).

Quantities (7.1) are precisely those quantities that allow to satisfy, *period after period*, final demand for consumption goods, while keeping labour force and productive capacity full employed, and therefore providing for those new investments, according to conditions (4.14), that maintain capital accumulation in line with the evolution of effective demand.

Prices (7.2) are the other side of the coin: they are precisely those *exchange ratios* which, given the distributive variables, are *necessary* for the economic system to produce, period after period, exactly equilibrium quantities (7.1).

To be more precise, what we have is a whole *set* of equilibrium configurations of relative prices and relative quantities, one for each possible combination of the distributive variables. The next step of the analysis, to catch up Pasinetti (1981), consists in choosing one of these configurations, and specifically the one defining the 'natural' economic system. That is to say, the 'natural' economic system is given by expressions (7.1) and (7.2) closed by means of a particular *theory of distribution*, giving a particular combination of the rate(s) of profit and therefore, given the *numéraire*, of the wage rate.

The aim of this section is that of sketching such a theory of income distribution and the main consequences and implications of its adoption as the closure of the price system.

## 9.1 'Natural' rates of profit and 'natural' prices

The theory of income distribution underlying Pasinetti's (1981) approach is discussed in detail in Garbellini & Wirkierman (2010b, section 4.1). Suffice here to recall that Pasinetti's purpose is "to develop first of all a theory which remains neutral with respect to the institutional organisation of society" (Pasinetti 1981, p. 25). In order to close the price system according to such a purpose, he had to find out a way of treating income distribution independently of institutional considerations.

How is it possible to do so, when "the way in which income is distributed crucially depends on the character of the *social relations of production*, no less than on cultural, ethic, legal considerations, that is to say, precisely on the institutional set-up of society" (Garbellini & Wirkierman 2010b, section 4.1, p. 21)?

The main idea is that of attributing to wages and profits two different *functions*: while the former provide for the purchasing power which must absorb the production of *consumption commodities*, i.e. those commodities *not re-entering the circular flow*, the latter must provide for the purchasing power necessary for ensuring equilibrium *capital accumulation*, i.e. for absorbing *new investment* commodities which, on the contrary, *do re-enter the circular flow*.

In few words, Pasinetti states a theory of *functional income distribution*, according to which each vertically hyper-integrated sector $i$ ($i = 1, 2, \ldots, m$) has its own 'natural' rate of profit, exactly equal to its own specific rate of growth: $g + r_i$.

Therefore, the 'natural' rates of profit to be used to close the price system are given by:

$$\pi^n_{i,t} = g + r_{i,t+1} = c_{i,t+1}, \quad i = 1, 2, \ldots, m \tag{9.1}$$

The main difference with respect to Pasinetti (1981) already emerged in Pasinetti (1988): when we close the price system with the natural rates of profit, we clearly do not have a uniform rate of profit anymore, but a whole series of $m$ *sectoral* rates of profit $\pi^n_{i,t}$, $i = 1, 2, \ldots, m$. Therefore, we also have $m$ 'natural' price systems, one for each vertically hyper-integrated sector:

$$\begin{cases} \mathbf{p}_{i,t}^{nT} = \overline{w}\mathbf{z}_t^{(i)T} \\ \mathbf{p}_{k_i,t}^{nT} = \overline{w}\mathbf{z}_t^{(i)T}\mathbf{M}^{(i)} \end{cases} \qquad i = 1, 2, \ldots, m \tag{9.2}$$

or, when embodied labour is adopted as the *numéraire* commodity for the price system, and therefore we set $\overline{w} = 1$:

$$\begin{cases} \mathbf{p}_{i(w),t}^{(i)nT} = \mathbf{z}_t^{(i)T} \\ \mathbf{p}_{k_i(w),t}^{nT} = \mathbf{z}_t^{(i)T}\mathbf{M}^{(i)} \end{cases} \qquad i = 1, 2, \ldots, m \tag{9.3}$$

Each vertically hyper integrated sector $i$ is therefore characterised by a specific 'natural' price system. The first line of expression (9.2) — or, equivalently, of expression (9.3) — gives the prices of *all* the $m$ commodities produced in the sector *in ordinary units*. The $i$-th element of vector $\mathbf{p}_{i,t}^{nT}$ $(i = 1, 2, \ldots, m)$ — or $\mathbf{p}_{i(w),t}^{nT}$ — is the 'natural' price of *consumption commodity* $i$, i.e. the final consumption commodity defining the vertically hyper-integrated sector. The other $m-1$ elements are defined by Pasinetti (1988) 'hypothetical' prices: *if* all the $m$ commodities produced in the economic system as a whole were produced *as consumption commodities* in vertically hyper-integrated sector $i$, these would be the corresponding prices. In fact however (as explained in detail in Garbellini 2010b, sections 5.3 and 6) they can also be seen as the prices, in ordinary units, of the single commodities entering vertically hyper-integrated productive capacity, and therefore used to compute the corresponding prices $p_{k_j,t}^{(i)}$ $(i, j = 1, 2, \ldots, m)$.

The whole vector is thus used in the second line of expression (9.2) — or, equivalently, of expression (9.3) — to compute the price of the units of vertically hyper-integrated productive capacity. Also here, the relevant element of vector $\mathbf{p}_{k_i,t}^{nT}$ — or $\mathbf{p}_{k_i(w),t}^{nT}$ — is the $i$-th one. The other elements would be the prices of the units of productive capacity for consumption commodities $j \neq i$, were they produced in vertically hyper-integrated sector $i$ as final consumption commodities. However, exactly as for the elements of vector $\mathbf{p}_{i,t}^{nT}$ other than the $i$-th one, they can also be seen as the prices of each single component of vertically hyper-integrated productive capacity for producing one unit of productive capacity itself, and are therefore those prices that we would use in case we wanted to compute the price of such 'higher order' productive capacity: $\mathbf{p}_{k_i,t}^{nT}\mathbf{M}^{(i)}$.

The just given definition of the 'natural prices' immediately implies a reflection on the relation between the fundamental and the institutional stages of the analysis:

> after developing our analysis independently of institutions, it may well emerge that some of the 'natural' features of an economic system may be

> impossible to achieve within a particular institutional set-up. In fact, the foregoing analysis precisely points at the 'natural rates of profit' as a most clear example of this type of impossibility.
>
> (Pasinetti 1981, p. 151)

That is to say, Pasinetti states, the 'natural' configuration of income distribution, and therefore of prices, is impossible to be achieved within a capitalist economic system. This impossibility is even more clear here — as well as in Pasinetti (1988) — than in Pasinetti (1981), due to the re-introduction of the more general description of the technique in use.

In the simplified formulation adopted by Pasinetti (1981), in fact, any commodity is produced *either* as a consumption good, *or* as an intermediate commodity; moreover, each intermediate commodity is utilised as an input by only one specific sector, and does not have any role in the others. The price of a consumption commodity simply depends on the amount of labour necessary for its production and on the price of its own 'capital' good; thus, the introduction of the (non-uniform) 'natural' rates of profit, as long as we keep institutional considerations outside the picture, does not create too many complications, since interactions between different industries are ruled out. The only incompatibility with a capitalist social mode of production is that in such an institutional framework the rate of profit is generally thought of as being uniform; in the 'natural' economic system this could only happen in presence of uniform rates of change of demand for final consumption commodities, which clearly is quite an irrealistic option.

In the general case, things are much more complicated. As it appears from expressions (9.2) and (9.3), when we use the complete matrix $\mathbf{A}$, and then close the price system with the 'natural' rates of profit, we get *a whole series* of $m$ 'natural' price systems, one for each vertically hyper-integrated sector $i$ $(i = 1, 2, \ldots, m)$. This means that each commodity produced in the economic system as a whole has only one natural price as a *consumption good*, and $m$ different natural prices as an *intermediate commodity*, in ordinary units, according to the specific vertically hyper-integrated sector whose corresponding productive capacity it is part of. Indeed, this is something more than non-uniformity in the rate of profit of different *industries*; since one single industry enters more than one vertically hyper-integrated sector — $m$, if it produces a basic commodity — and since all the activities participating in the same vertically hyper-integrated sector do charge the same rate of profit, we should have different rates of profit *within the very same industry*. Clearly, this is at odds with what we can actually observe in any capitalist economic system.

More specifically, if we consider vertically hyper-integrated sector $i$ with non-uniform rates of profit $\pi_{i,t}$, the corresponding prices are given by

$$\begin{cases} \mathbf{p}_t^{(i)T} = \overline{w}\mathbf{z}_t^{(i)T}\left(\mathbf{I} - \mathbf{M}^{(i)}(\pi_{i,t} - c_{i,t+1})\right)^{-1} \\ \mathbf{p}_{k,t}^{(i)T} = \overline{w}\mathbf{z}_t^{(i)T}\mathbf{M}^{(i)}\left(\mathbf{I} - \mathbf{M}^{(i)}(\pi_{i,t} - c_{i,t+1})\right)^{-1} \end{cases} \tag{9.4}$$

which, when $\pi_{i,t} = \pi_{i,t}^n$ reduces to:

$$\begin{cases} \mathbf{p}_t^{(i)nT} = \overline{w}\mathbf{z}_t^{(i)} \\ \mathbf{p}_{k,t}^{(i)nT} = \overline{w}\mathbf{z}_t^{(i)T}\mathbf{M}^{(i)} = \overline{w}\mathbf{z}_{k,t}^{(i)T} \end{cases} \tag{9.5}$$

Since there are $m$ expression like (9.4), or (9.5) — one for each vertically hyper-integrated sector — each commodity $j$ $(j = 1, 2, \ldots, m)$ has one price when considered as the consumption good produced by vertically hyper-integrated sector $j$— specifically, the $j$-th component of the corresponding price vector $\mathbf{p}_t^{(j)T}$, or $\mathbf{p}_t^{(j)nT}$ — and one price when considered as an intermediate commodity for each vertically hyper-integrated sector $i$, that is to say when it is part of its vertically hyper-integrated productive capacity — specifically, the $j$-th component of the corresponding price vector $\mathbf{p}_t^{(i)T}$ or $\mathbf{p}_t^{(i)nT}$, with $i = 1, 2, \ldots, m$.

Expression (9.5) also makes another characteristic of the 'natural' economic system come to the fore. We can in fact see that the 'natural' price of each consumption commodity, and of each unit of vertically hyper-integrated productive capacity, is given by the product of the wage rate and the corresponding vertically hyper-integrated labour coefficient. The implication is straightforward: thanks to the redefinition of the concept of net output, labour embodied is thought of not only as direct and indirect, but also ad *hyper-indirect* labour. Together with the adoption of the particular theory of income distribution sketched at the beginning of the present section, and implying the 'natural' rates of profit, it is then possible to go back to a *pure labour theory of value*, even when a rate of profit does exist and when intermediate commodities are considered. All 'natural' profits immediately translate into new investments, and thus into wages for those labourers producing new investment commodities; the 'natural' price of each commodity is exactly equal to its 'labour value', labour embodied and labour commanded thus coming to coincide.

To conclude this section, we may now go back to the analysis of section 6, to look more in detail at the dynamics of the real wage rate within the 'natural' economic system, i.e. when 'natural' rates of profit are adopted as the closure of the price system.

In this case, if *consumption* commodity $h$ is chosen as the *numéraire* of the price system(s), its price will be kept fixed to 1 in all time periods; therefore, its rate of change through time will be $\sigma_{p_h,t} = \sigma_{p_h} = 0$. Hence, the real wage rate — and its rate of change through time — in terms of commodity $h$ will be:

$$\begin{cases} w_t^{(h)} = \dfrac{1}{z_{h,t}^*}, \quad \forall t \\ \\ \sigma_{w_t}^{(h)} = \varrho'_{h,t} \end{cases} \tag{9.6}$$

Any increase/decrease in the vertically hyper-integrated labour coefficient for the commodity whose price is chosen as the *numéraire* immediately translates into an equal decrease/increase in the real wage rate. Therefore, all increases in labour productivity immediately translate into a corresponding increase in labourers' real purchasing power: all price reductions made possible by an increased labour productivity are gained by labourers themselves, instead of being (partially) absorbed by profits.

## 9.2 Sectoral capital/output and capital/labour ratios in the 'natural' economic system

It is worth spending a few words on the meaning that capital/output and capital/labour ratios come to acquire within the 'natural' economic system.

First of all, there is a major difference with respect to the sectoral capital/output ratios computed for the general case; when evaluated at current prices different from the 'natural' ones, the stock of capital available at the beginning of the production process is given by the product of the wage rate and the corresponding labour *equivalent*; the same holds for sectoral output. As a consequence, the sectoral capital/output ratios are ratios of two quantities of labour equivalent. On the contrary, within the 'natural' economic system they are ratios of *physical quantities of labour*, i.e.:

$$\gamma_{i,t}^n = \frac{p_{k_i,t}^n k_{i,t}}{p_{i,t}^n x_{i,t}} \equiv \frac{\mathbf{z}_t^{(i)T}\mathbf{m}_i^*}{z_{i,t}^*} = \frac{\mathbf{z}_{t-1}^{(i)T}(\mathbf{I} - \widehat{\boldsymbol{\varrho}}_t^{(i)\prime})\mathbf{m}_i^*}{z_{i,t-1}^*(1-\varrho'_{i,t})} \tag{9.7}$$

or

$$\gamma_{i,t}^n = \frac{z_{k_i,t}^*}{z_{i,t}^*} = \frac{z_{k_i,t-1}^*(1-\varrho'_{k_i,t})}{z_{i,t-1}^*(1-\varrho'_{i,t})} \tag{9.8}$$

where

$$\varrho'_{k_i,t} = -\frac{\mathbf{z}_t^{(i)T}\mathbf{m}_i^* - \mathbf{z}_{t-1}^{(i)T}\mathbf{m}_i^*}{\mathbf{z}_{t-1}^{(i)T}\mathbf{m}_i^*} = -\frac{\mathbf{z}_{t-1}^{(i)T}(\mathbf{I} - \widehat{\boldsymbol{\varrho}}_t^{(i)\prime})\mathbf{m}_i^* - \mathbf{z}_{t-1}^{(i)T}\mathbf{m}_i^*}{\mathbf{z}_{t-1}^{(i)T}\mathbf{m}_i^*} = \frac{\mathbf{z}_{t-1}^{(i)T}\widehat{\boldsymbol{\varrho}}_t^{(i)\prime}\mathbf{m}_i^*}{\mathbf{z}_{t-1}^{(i)T}\mathbf{m}_i^*} \tag{9.9}$$

where $\varrho_{k_i,t}^{(i)}$ is the rate of change from $t$ to $t+1$ of the vertically hyper-integrated labour necessary for the production of one unit of productive capacity for vertically hyper-integrated sector $i$ ($i = 1, 2, \ldots, m$), $\mathbf{z}^{(i)T}\mathbf{m}_i^*$.

The rate of change from $t$ to $t+1$ of such ratios is given by:

$$\sigma_{\gamma_{i,t}^n} = \frac{\mathbf{z}_{t-1}^{(i)T}(\varrho_{i,t}'\mathbf{I} - \widehat{\boldsymbol{\varrho}}_t^{(i)\prime})\mathbf{m}_i^*}{\mathbf{z}_{t-1}^{(i)T}(1 - \varrho_{i,t}')\mathbf{m}_i^*} \tag{9.10}$$

or:

$$\sigma_{\gamma_{i,t}^n} = \frac{\varrho_{i,t}' - \varrho_{k_i,t}'}{1 - \varrho_{i,t}'} \tag{9.11}$$

which means that the 'natural' capital output ratio — i.e. the ratio of vertically hyper-integrated labour embodied in one unit of vertically hyper-integrated productive capacity to that embodied in one unit of final output — increases when (vertically hyper-integrated) labour increases in the production of consumption commodities more than in the production of the corresponding units of productive capacity.

The degree of mechanisation of vertically hyper-integrated sector $i$ ($i = 1, 2, \ldots, m$) is the ratio between the *stock* of capital, evaluated at current prices, and the *flow* of labour employed in the production process:

$$\theta_{i,t}^n = \frac{p_{k_i,t}^n k_{i,t}}{z_{i,t}^* x_{i,t}} = \frac{p_{k_i,t}^n x_{i,t}}{z_{i,t}^* x_{i,t}} = \frac{w_t \mathbf{z}_t^{(i)T}\mathbf{m}_i^*}{z_{i,t}^*} = \frac{w_{t-1}\mathbf{z}_{t-1}^{(i)T}\left(\mathbf{I}(1 + \sigma_{w,t}^{(h)}) - \widehat{\boldsymbol{\varrho}}_t^{(i)\prime}\right)\mathbf{m}_i^*}{z_{i,t-1}^*(1 + \varrho_{i,t}')} \tag{9.12}$$

or:

$$\theta_{i,t}^n = \frac{w_t z_{k_i,t}^*}{z_{i,t}^*} = \frac{w_{t-1} z_{k_i,t-1}^*(1 + \sigma_{w,t}^{(h)} - \varrho_{k_i,t}')}{z_{i,t-1}^*(1 - \varrho_{i,t}')} \tag{9.13}$$

its rate of change through time being:

$$\sigma_{\theta_{i,t}^n} = \frac{\mathbf{z}_{t-1}^{(i)T}\left(\mathbf{I}(\sigma_{w,t}^{(h)} - \varrho_{i,t}') - \widehat{\boldsymbol{\varrho}}_{k_i,t}^{(i)\prime}\right)\mathbf{m}_i^*}{\mathbf{z}_{t-1}^{(i)T}(1 - \varrho_{i,t}')\mathbf{m}_i^*} \tag{9.14}$$

or

$$\sigma_{\theta_{i,t}^n} = \frac{\sigma_{w,t}^{(h)} - \varrho_{k_i,t}' + \varrho_{i,t}'}{1 - \varrho_{i,t}'} \tag{9.15}$$

Expression (9.15) tells us that the 'natural' degree of mechanisation increases when (vertically hyper-integrated) labour productivity in the production of the units of productive capacity is greater than the sum of labour productivity increases in the production of the corresponding consumption commodity and the rate of growth of the wage rate — that of course is different according to which commodity $h$ ($h = 1, 2, \ldots$) is chosen as the *numéraire* of the price system.

With respect to the dynamics of the 'natural' capital intensity, therefore, here the role played by the movements of the wage rate is therefore apparent. Within the 'natural' economic system, such ratios change because of changes in the quantity of labour *embodied in the units of productive capacity* and because of changes in the wage rate (which of course are influenced by the choice of the *numéraire* too). Therefore, capital intensity and degree of mechanisation change in the same direction, and actually in the very same proportion, only when the wage rate is constant, and therefore the second effect — which is specific to capital/labour ratios — disappears.

In all other cases, their movements through time do not go, in principle, in the same direction. Differences in their trajectories are not predictable, and the capital/labour ratios cannot be taken as indicators of capital intensity.[12]

## 9.3 The 'standard rate of growth of productivity' and the 'dynamic standard commodity'

As a further characteristic of the 'natural' economic system, Pasinetti (1981, Chapter V, sections 13-14) introduces the 'standard rate of growth of productivity', and hence the 'dynamic standard commodity', which then he uses — among the other things — to define the concept of 'natural' rate of interest. In a few words, the basic original idea is that the 'standard rate of growth of productivity' is a weighted average of the rates of growth of labour productivity in the various sectors, which can also be seen as the rate of growth of productivity of a particular composite commodity — the 'dynamic standard commodity' — which, if used as the *numéraire* of the price system, possesses the remarkable property of keeping the average price level constant through time.

Let us first of all briefly summarise the analytical formulation of the 'standard rate of growth of productivity' originally put forward by Pasinetti (1981), though with the modification of considering discrete, rather than continuous, time.

According to Pasinetti (1981), the 'standard rate of growth of productivity' is a *weighted average* of the rates of growth of productivity in the $m$ vertically hyper-integrated sectors composing the economic system as a whole, the weights

[12]For details about the consequences of doing so, see Pasinetti (1981, Chapter IX, section 7).

being $\lambda_{i,t}$ $(i = 1, 2, \ldots, m)$:

$$\varrho_t^* = \sum_{i=1}^{m} \lambda_{i,t} \varrho'_{i,t} \tag{9.16}$$

The problem is now that of finding the proper weights to compute this average; Pasinetti solves the problem by observing that the addenda entering the macroeconomic condition:

$$\sum_{i=1}^{m} a_{in,t} z_{i,t}^* = 1 \tag{9.17}$$

besides adding up to 1 — which is of course a necessary requirement for them to be the weights to be used to compute a weighted average — represent the *proportion* of total labour required by the each vertically hyper-integrated sector. Hence, they are precisely the weights we are looking for:

$$\varrho_t^* = \sum_{i=1}^{m} \lambda_{i,t} \varrho'_{i,t} = \sum_{i=1}^{m} \left( a_{in,t} z_{i,t}^* \right) \varrho'_{i,t} \tag{9.18}$$

By having a closer look at the $\lambda_i$'s, by direct examination of the macroeconomic condition, written as:

$$\sum_{i=1}^{m} z_{i,t}^* a_{in,t} = \sum_{i=1}^{m} \lambda_{i,t} = 1 \tag{9.19}$$

we can stress first of all that such weights change themselves through time, and are therefore in principle different from period to period. Moreover, each $\lambda_{i,t}$ can be seen in two ways:

(i) as the proportion of total labour employed by vertically hyper-integrated sector $i$ — also when we are not within the 'natural' economic system; and
(ii) *only within* the 'natural' economic system, as the proportion of the total wages spent for buying consumption commodity $i$.

Therefore, $\varrho_t^*$ can be seen as the rate of change in the vertically hyper-integrated labour coefficient, from time period $t-1$ to $t$, of a hypothetical (composite) sector producing a particular *composite commodity* — the 'dynamic standard commodity'. Let us call $z_{s,t}$ such a (composite) labour coefficient (where the subscript $s$ stands for 'standard' commodity).

We can now go back to the beginning of the present section, recalling the main property of the 'dynamic standard commodity': when it is used as the *numéraire* of the price system, the (average) price level remains constant through time. Pasinetti (1981) exploits this property, as already hinted at above, to arrive to the definition of the 'natural' rate of interest,[13] and specifically to express the 'real' and 'nominal' rate of inflation.

It is therefore clear that when Pasinetti refers to the average price level, to be kept constant in terms of the 'dynamic standard commodity' — whose composition, it is worth stressing, changes through time — he is referring to a magnitudes which is related to the *purchasing power* of the average consumer, since its dynamics is at the basis of the idea of *price inflation*.

The purchasing power of the average consumer — or better, of his/her income — depends not only on (relative) prices, but also by the basket of goods he/she wants, or needs, to consume in every specific point in time. That is to say, when we try to compute changes in the individuals' purchasing power, we do not care about the absolute changes of prices, but about the interaction of such changes with the shifts of the composition of final demand. If the price of a certain consumption commodity undergoes a great increase relatively to those of the others, but it is a very small fraction, in the final period, of the actually consumed basket of goods, the effect on the change in households real purchasing power will be very small indeed.

With this idea in mind, we clearly cannot but define the purchasing power of the average consumer — or the average wage earner, which within the 'natural' economic system is precisely the same — at time $t$ as the real value of the wage rate in terms of the specific basket of goods actually consumed, i.e. the composite commodity $[a_{1n,t} \ a_{2n,t} \ \dots \ a_{mn,t}]$. Saying that the purchasing power of the wage rate is constant through time thus amount at saying that *the real wage rate*, expressed in terms of such a composite commodity, is constant through time.

In terms of whatever *numéraire* we may arbitrarily choose for the price system, in analytical terms such a condition can be written as:

$$\frac{w_t}{\sum_{i=1}^{m} p_{i,t}^{*} a_{in,t}} = \frac{w_t}{w_t \sum_{i=1}^{m} z_{i,t}^{*} a_{in,t}} = \frac{1}{\sum_{i=1}^{m} z_{i,t}^{*} a_{in,t}} = 1, \quad \forall t \tag{9.20}$$

Hence, within the 'natural' economic system, the real purchasing power of the wage rate, defined as we did define it above, is *always* constant, whatever the *numéraire* we choose for the price system.

This is an interesting conclusion, revealing another feature of the 'natural' economic system: in real terms, with respect to the basket of goods actually composing final demand for consumption commodities, the real purchasing power of the wage rate is constant through time.

But yet, up to now we have been talking of the purchasing power of the wage rate, not of the average price level. Consistently with the definition of purchasing

[13]See section 11.

power itself, the average price level for time period $t$ can be defined as the price of the basket of goods actually consumed by the average consumer:

$$\overline{p}_t^* = \sum_{i=1}^{m} p_{i,t}^* a_{in,t} = w_t \sum_{i=1}^{m} z_{i,t}^* a_{in,t} \tag{9.21}$$

We now want to see whether the 'dynamic standard commodity', as defined by Pasinetti (1981), keeps the average price level constant through time when used as the *numéraire* commodity.

As we have seen at the end of the previous section, if we want to use such a commodity as the *numéraire* of the price system, we do not need to know its composition; we simply have to express the wage rate, both in a specific point in time and through time, in terms of it, and inserting the resulting expression into the price system. This would amount at setting $p_{s,t}^* = w_t z_{s,t} = 1$, and therefore:

$$\begin{cases} w_t^{(s)} = \dfrac{1}{z_{s,t}^*} \\ \\ \sigma_{w_t}^{(s)} = \varrho_t^* \end{cases} \tag{9.22}$$

When the 'dynamic standard commodity' is used as the *numéraire* of the price system, any decrease in total (vertically hyper-integrated) labour necessary for its production translates into a proportional increase in the real wage rate. It is important to stress that we can use such a commodity as the *numéraire* even *without knowing its composition*.

Moreover, when the 'dynamic standard commodity' is used as the *numéraire* of the price system, the rate of change of the price of any commodity $i$ ($i = 1, 2, \ldots, m$) is given by:

$$\sigma_{p_{i,t}}^{(s)} = \varrho_t^* - \varrho_{i,t}' \tag{9.23}$$

i.e. by the difference between the 'standard rate of growth of productivity' and the rate of change of the corresponding vertically hyper integrated labour coefficient. This means that the 'natural' price of a commodity increases when the vertically hyper-integrated labour productivity is smaller than the (weighted) average, and decreases when it is higher.

Let us now go back to expression (9.21), and compute its changes through time when the price of the 'dynamic standard commodity' is the *numéraire* of the price system:

$$\overline{p}_t^* - \overline{p}_{t-1}^* = w_{t-1}(1+\varrho_t^*)\sum_{i=1}^{m} z_{i,t-1}^* a_{in,t-1}(1-\varrho_{i,t}'+r_{i,t}) - w_{t-1}\sum_{i=1}^{m} z_{i,t-1}^* a_{in,t-1} =$$
$$= w_{t-1}\sum_{i=1}^{m} z_{i,t-1}^* a_{in,t-1}(\varrho_t^* - \varrho_{i,t}' + r_{i,t})$$

In order for this change to be equal to zero, i.e. for the average price level to be constant through time, the following condition should be holding:

$$w_{t-1}\sum_{i=1}^{m} z_{i,t-1}^* a_{in,t-1}(\varrho_t^* - \varrho_{i,t}' + r_{i,t}) = 0$$

i.e.:

$$\varrho_t^* = \sum_{i=1}^{m} z_{i,t-1}^* a_{in,t-1}(\varrho_{i,t}' - r_{i,t}) = \sum_{i=1}^{m} \lambda_{i,t}(\varrho_{i,t}' - r_{i,t}) \tag{9.24}$$

We immediately notice the difference with respect to Pasinetti's (1981) original formulation: the 'standard rate of growth of productivity', as it emerges from our formulation, is a weighted average — the weights being the $\lambda_{i,t}$s — not of the rates of growth of labour productivity in the $m$ vertically hyper-integrated sectors, but of the difference between such rates and the rate of growth of final per-capita demand for the corresponding consumption commodity.

This conclusion is the quite obvious consequence of what stated above: changes in the purchasing power, and thus in the average price level, do depend not only on the variations of (relative) prices, but also, and in a very relevant way, for the variations in the composition of final demand. Therefore, a 'standard rate of growth of productivity' has to take into account both determinants.

I may add that this reformulation allows us to overcome a further difficulty connected with Pasinetti's (1981) original definition of the 'standard rate of growth of productivity' and thus of the 'dynamic standard commodity'. The latter do make sense *only* within the natural economic system, where prices and labour values do coincide, the weights $\lambda_{i,t}$s do have the double meaning mentioned above, and therefore the rate of change of the real purchasing power do coincide with the rate of change of labour productivity in the vertically hyper-integrated sector producing the *numéraire* (composite) commodity. More specifically, Pasinetti follows the following reasoning: prices change because of changes, and exactly in the same proportion as, labour requirements change. Therefore, if we choose the weighted average of the rates of change of labour productivity in the different vertically hyper-integrated sectors obtained by using the $\lambda_{i,t}$s as the weights — which are both the proportion of sectoral to total labour, but also of the average per-capita income spent for buying consumption commodity $i$ with respect to the total — as the *numéraire* of the price system, one half of the prices will increase, and the other half decrease, on *weighted* average, the positive and the negative changes therefore canceling out. Clearly, this reasoning only holds *within* the 'natural' economic system. But as soon as we consider prices different from the natural ones, their changes are not caused by and exactly in the same proportion as changes in vertically hyper-integrated labour requirements, but are caused by and exactly in the same proportion as changes in the corresponding labour equivalents. Changes in distributive variables, and not only in labour productivity, come to affect the average price level.

On the contrary, the reformulation given in this section do make sense both within and outside the 'natural' economic system, since it implies the definition of the 'dynamic standard commodity' as a composite commodity that, whatever the rates of profit, if used as the *numéraire* of the price system keeps, *by definition*, the average price level constant through time. The 'standard rate of growth of productivity', outside the natural economic system, is not to be seen as the rate of change of the real purchasing power of the wage rate when the 'dynamic standard commodity' is adopted as the *numéraire* of the price system: such a rate of change is given by the associated reduction in the corresponding *labour equivalent*;[14] anyway, it is precisely the rate of change of labour productivity in the associated, hypothetical (composite) vertically hyper-integrated sector producing the 'dynamic standard commodity' itself.

## 10 The 'natural' price system through time: choice of the *numéraire* and rate of inflation

All the magnitudes considered, and analysed, so far are *real* magnitudes, i.e. magnitudes whose value is expressed in terms of some physical commodity, or composite commodity, or of labour. A *conventional unit of account*, such as paper money, needs to be introduced in order to be able of treating *nominal* magnitudes; at the

[14]To be more precise, this rate of change, call it $\sigma_{w,t}^{(s)}$, where the superscript $s$ stands for *standard* commodity, is given by:

$$\sigma_{w,t}^{(s)} = \frac{\mathbf{z}_{t-1}^{(e)T}\left(\widehat{\boldsymbol{\varrho}}_{t-1}^{(e)\prime} - \widehat{\mathbf{r}}_{t-1}\right)\mathbf{a}_{in,t-1}}{\mathbf{z}_{t-1}^{(e)T}\mathbf{a}_{in,t-1}}$$

same time, a way of establishing a relation between nominal and real magnitudes is necessary in order to attach a concrete meaning to the former.

When paper money is used as the basis for the price system, expressing commodity prices in terms of it requires to close the same two degrees of freedom left open in the (relative) commodity price system, exactly in the same way as we did when choosing any physical commodity as the *numéraire*. that is to say, we have to arbitrarily fix the initial value, and the rate of change through time, of the wage rate in terms of money, by setting:

$$\begin{cases} w_0 = w_0^{(M)} \\ \sigma_{w_t} = \sigma_{w_t}^{(M)} \end{cases} \tag{10.1}$$

As Pasinetti (1981, p. 162) points out, not only $w_0^{(M)}$, but also $\sigma_{w_t}^{(M)}$ can be arbitrarily fixed at *any* level; nonetheless, such a rate of change would be a purely nominal one, as long as we do not give it a physical content by establishing a relation between it and real magnitudes.

Let us now adopt such a conventional unit of account as the *numéraire* of the price system, and compute the rate of change through time of the *nominal* (average) price level:

$$\begin{aligned} \sigma_{\overline{p}_t^{(M)*}} &= \frac{w_{t-1}^{(M)}\left((1+\sigma_{w_t}^{(M)})\sum_{i=1}^m \lambda_{i,t-1}(1+r_{i,t}-\varrho'_{i,t}) - \sum_{i=1}^m \lambda_{i,t-1}\right)}{w_{t-1}^{(M)}\sum_{i=1}^m \lambda_{i,t-1}} = \\ &= \sigma_{w_t}^{(M)} - \sum_{i=1}^m \lambda_{i,t-1}(\varrho'_{i,t} - r_{i,t}) = \sigma_{w_t}^{(M)} - \varrho_t^* \end{aligned} \tag{10.2}$$

It immediately appears from expression (10.2) that an *ideal situation* of *nominal* (average) *price stability* — that can be taken as a reference point — would occur in the special case in which $\sigma_{w_t}^{(M)} = \varrho_t^*$.

Any time that $\sigma_{w_t}^{(M)} > \varrho_t^*$, the general nominal price level would be increasing, and therefore we would be in a situation of *price inflation*; by contrast, any time that $\sigma_{w_t}^{(M)} < \varrho_t^*$ the nominal price level is decreasing, and we are in a situation of *price deflation*.

Accordingly, we can therefore define the *general level of price inflation* ($\sigma_{A,t}^{(M)}$) as:

$$\sigma_{A,t}^{(M)} = \sigma_{w_t}^{(M)} - \varrho_t^*$$

and hence write the rate of change through time of the 'natural' price of any (consumption) commodity $i$ ($i = 1, 2, \ldots, m$) as:

$$\sigma_{p_{i,t}^*}^{(M)} = \sigma_{w_t}^{(M)} - \varrho'_{i,t} \equiv \left(\sigma_{w_t}^{(M)} - \varrho_t^*\right) + \left(\varrho_t^* - \varrho'_{i,t}\right) \equiv \sigma_{A,t}^{(M)} + \left(\varrho_t^* - \varrho'_{i,t}\right) \tag{10.3}$$

As it is clear from expression (10.3), the rate of change through time of the price of a commodity is made up by two components: the general level of price inflation, affecting *all* prices, over and above the specific changes in labour productivity; and a *sector-specific* component, namely $(\varrho_t^* - \varrho'_{i,t})$, affecting only the price of the commodity produced in the corresponding vertically hyper-integrated sector.

## 11 The 'natural' rate of interest

We are now in the position to move — step by step, as Pasinetti (1981) does — towards the definition of a 'natural' rate of interest. Doing so implies introducing into the picture a whole set of assets and liabilities, i.e. of debt and credit relations to be stipulated between individuals, or group of individuals, and that cancel out at the aggregate level.

As Pasinetti notices,

> [t]he immediate consequence of the introduction of financial assets and liabilities into our analysis is that it becomes no longer indifferent which commodity is chosen as the *numéraire* of the price system [...]. For, the choice of the *numéraire* ties down all debts and credit to being constant through time in terms of the particular commodity chosen as the *numéraire*; while, at the same time, all 'natural' prices are changing in terms of that *numéraire*. (Pasinetti 1981, p. 158)

To see how this happens, let us first suppose to choose the price of commodity $h$ as the *numéraire* of the price system. Be $i^{(h)}$ the interest rate stipulated between the borrower and the lender on the amount of the loan; consider a loan stipulated at time $t$ — with a *zero rate of interest* — and expiring at time $t+1$.

Since $h$ is the *numéraire* commodity, the loan is stipulated in terms of it, which means that a certain amount of purchasing power, in terms of commodity $h$, as been lent at time $t$, and *exactly the same amount* of purchasing power in terms of the *numéraire* must be given back at time $t+1$, at expiration of the loan. Clearly, such a purchasing power is kept constant through time in terms of the *numéraire* commodity, but will not be constant also in terms of all other commodities (or composite commodities).

In particular, let us consider commodity $i$ $(i = 1, 2, \dots, m)$. As usual, $\varrho'_{i,t}$ is the rate of change of productivity in vertically hyper-integrated sector $i$, and $\varrho'_{h,t} - \varrho'_{i,t}$ is the rate of change of the ('natural') price of commodity $i$ itself when the *numéraire* is the price of commodity $h$. Be $q_{i,t}^{(h)}$ the number of units of commodity $i$ that could be bought, with the amount of the loan, at time $t$. This means that the original value of the loan, at current prices, in terms of commodity $i$ is given by:

$$p_{i,t}^{(h)*}q_{i,t}^{(h)} = p_{i,t}^{(h)*}(1+\varrho'_{h,t}-\varrho'_{i,t})(1-\varrho'_{h,t}+\varrho'_{i,t})q_{i,t}^{(h)} \equiv p_{i,t+1}^{(h)*}q_{i,t+1}^{(h)}$$

and therefore:

$$q_{i,t+1}^{(h)} = (1-\varrho'_{h,t}+\varrho'_{i,t})q_{i,t}^{(h)} \tag{11.1}$$

Whenever $\varrho'_{i,t} > \varrho'_{h,t}$, the purchasing power in terms of commodity $i$ is greater at the expiration of the loan than at time $t$: the loan has undergone a *revaluation*, in terms of commodity $i$, at the rate $\varrho'_{i,t}-\varrho'_{h,t}$. On the contrary, whenever $\varrho'_{i,t} < \varrho'_{h,t}$, the purchasing power has decreased from $t$ to $t+1$, and therefore the loan, in terms of commodity $j$, has undergone a *devaluation* at the rate $\varrho'_{h,t}-\varrho'_{i,t}$.

Similarly, if the amount of the loan at time $t$ allowed to command $x_n^{(h)}$ hours of labour, its real value, in terms of labour, at time $t$ is given by:

$$w_t^{(h)}x_{n,t}^{(h)} = w_t^{(h)}(1+\varrho'_{h,t})(1-\varrho'_{h,t})x_{n,t}^{(h)} \equiv w_{t+1}^{(h)}x_{n,t+1}^{(h)}$$

and therefore

$$x_{n,t+1}^{(h)} = (1-\varrho'_{h,t})x_{n,t}^{(h)}$$

Whenever $\varrho'_{h,t} > 0$ the loan undergoes a *revaluation* in terms of labour at the rate $\varrho'_{h,t}$; whenever $\varrho'_{h,t} < 0$ the loan undergoes a *devaluation* at the rate $-\varrho'_{h,t}$.

Let us now suppose that the wage rate is chosen as the *numéraire* of the price system, and that the conditions of the loan are exactly the same as those of the previous case. Consider again any commodity $(i = 1, 2, \ldots, m)$, recalling that in this case the rate of change of any price with respect to to labour is given by $-\varrho'_{i,t}$. The initial value of the loan is therefore given by:

$$p_{i,t}^{(w)*}q_{i,t}^{(w)} = p_{i,t}^{(w)*}(1-\varrho'_{i,t})(1+\varrho'_{i,t})q_{i,t}^{(w)} \equiv p_{i,t+1}^{(w)*}q_{i,t+1}^{(w)} \tag{11.2}$$

and therefore:

$$q_{i,t+1}^{(w)} = (1+\varrho'_{i,t})q_{i,t}^{(w)} \tag{11.3}$$

Hence, whenever the rate of change of productivity in vertically hyper-integrated sector $i$ is positive, the loan undergoes a *revaluation* at the rate $\varrho'_{i,t}$; while, whenever it is negative, it undergoes a *devaluation* at the rate $-\varrho'_{i,t}$.

In a few words,

> [...] the existence of financial assets and liabilities, when coupled with a structural dynamics of natural prices, implies the existence, not of one rate of interest, but a whole series of rates of interest. More precisely, it implies the existence of a particular own-rate of interest for each commodity.
>
> (Pasinetti 1981, p. 159)

To be more precise, we have *many* series of own-rates of interest, one for each *numéraire* commodity, or composite commodity, we might decide to choose for the price system.

Such own-rates of interest are implied by the very *structural dynamics* of commodity 'natural' prices, and are therefore *always* present, on all assets and liabilities, over and above whatever rate of interest that might be stipulated by the lender and the borrower on the loan itself, which adds up to them. Going back to the example in which $h$ is the *numéraire* commodity, if the rate of interest on the loan has been decided to be $i^{(h)} \neq 0$, then the series of own-rates of interest would be given by:

$$\begin{cases} \left(i^{(h)} + \varrho'_{i,t} - \varrho'_{h,t}\right) & \text{for commodity } i, \quad i = 1, 2, \ldots, m \\ \left(i^{(h)} + \varrho'_{i,t}\right) & \text{for labour} \end{cases} \tag{11.4}$$

Clearly, these are all *real* own-rates of interest; if we want to talk about *nominal* ones, we have to introduce paper money — or any other conventional unit of account — as the basis of the price system.

If all assets and liabilities are all stipulated in terms of paper money, with a nominal interest rate equal to $i^{(M)}$, then the series of own-rates of interest would be:

$$\begin{cases} \left(i^{(M)} - \sigma^{(M)}_{A,t} - (\varrho^*_t - \varrho'_{i,t})\right) & \text{for commodity } i, \quad i = 1, 2, \ldots, m \\ \left(i^{(M)} - \sigma^{(M)}_{A,t} - \varrho^*_t\right) & \text{for labour} \end{cases} \tag{11.5}$$

We can however consider a *special case*, i.e. the nominal own-rate of interest for the 'dynamic standard commodity':

$$\left(i^{(M)} - \sigma^{(M)}_{A,t} - (\varrho^*_t - \varrho^*_t)\right) = \left(i^{(M)} - \sigma^{(M)}_{A,t}\right) \tag{11.6}$$

> which, as we may see, is simply the difference between the money rate of interest and the rate of inflation. [...] It represents a sort of average 'real' rate of interest for the economic system as a whole. We may call it the 'standard' real rate of interest.
>
> (Pasinetti 1981, p. 165)

And what about the 'natural' rate of interest? Pasinetti (1981) states the problem in a very clear and effective way:

> A *whole structure* of rates of interest exists in any case, whatever the actual 'nominal' rate of interest (even if it were fixed at zero) and whatever the *numéraire* chosen as the basis of the price system. In other words, a whole structure of own-rates of interest — all of them 'real' rates of interest — is unavoidably inherent in the structural dynamics of relative prices.
>
> [...]
>
> [T]he problem to be solved — within the present theoretical framework, may be stated in the following manner. From the infinite number of possible levels of the actual rate of interest (and by implication of the structure of own-rates of interest), is there a particular one that may be called the 'natural' level of the rate of interest? (And, by implication is there a 'natural' level of the whole structure of the own-rates of interest?)
>
> (Pasinetti 1981, p. 166)

The answer Pasinetti gives to this question is as straightforward as the question itself:

> In an economic system in which all contributions to, and and benefits from, the production process are regulated on the basis of quantities of labour, the 'natural' rate of interest cannot but be a zero rate of interest in terms of labour.
>
> (Pasinetti 1981, p. 166)

Or, as he concludes at the end of the chapter:

> [w]e may well say that income is distributed according to a 'labour principle of income distribution'.
>
> (Pasinetti 1981, p. 169)

Or again, to put in another way, the main characteristic of the 'natural' economic system is the equivalence of labour embodied and labour commanded. This must hold not only within a single period of time, but also *through time*. When labour productivity increases, the wage rate increases proportionally. This means that — as we have shown above — if a certain amount of purchasing power can command, at time $t$ an amount of labour equal to $x_{n,t}^{(h)}$ ($h$ being whatever *numéraire* we have chosen for the price system), at time $t+1$ it will be able to command only a quantity $x_{n,t+1}^{(h)} = x_{n,t}^{(h)}(1 - \varrho'_{h,t})$. In order to restore the equivalence between labour embodied and commanded, therefore, the lent/borrowed amount of purchasing power must be 'augmented' through an interest rate — the *'natural' interest rate* — equal to $\varrho'_h$, that is equal to *the rate of growth of the wage rate, in terms of whatever numéraire is actually the basis of the price system.*

In short, in order to preserve *through time* the equivalence between labour embodied and labour commanded — i.e., in order to preserve the main characteristic of the 'natural' price system — the particular level of the rate of interest that may be called the 'natural' rate of interest is given by:

$$\begin{cases} \sigma_{w_t}^{(h)} & \text{if commodity } h \text{ is the } \textit{numéraire} \text{ commodity} \\ \sigma_{w_t}^{(s)} & \text{if the `dynamic standard commodity' is the } \textit{numéraire} \text{ commodity} \\ \sigma_{w_t}^{(M)} & \text{if paper money is the } \textit{numéraire} \text{ commodity} \\ \sigma_{w_t}^{(w)} = 0 & \text{if labour is the } \textit{numéraire} \text{ commodity} \end{cases}$$

## 12 Conclusions

As stated in the Introduction, the main task of the present paper was that of introducing in Pasinetti's (1981) and Pasinetti's (1988) original formulations — besides the complete description of the technique in use, already introduced in Garbellini (2010b) — discrete, rather than continuous, time, and thus non-steady rates of change of both sectoral (average) per-capita demand for consumption commodities and sectoral labour productivities. As explained in Garbellini (2010c, section 3.4), this aimed at increasing the realism of the whole formulation, at the same time making it more suitable for empirical analyses and allowing to see more in detail the dynamics of the main economic magnitudes of interest.

In particular, the introduction of the most general description of the technique in use in each time period — i.e., of the whole set of inter-industry relations — allowed to deepen, with respect to Pasinetti (1981), the analysis of the dynamics of commodity relative prices and of labour productivities.

As to the relative price system, the simplifying assumptions made in Pasinetti (1981) on the relations between the industries producing inputs and outputs were such as to make the price of each final consumption commodity $i$ ($i = 1, 2, \dots, m$) depend on the quantity of vertically hyper-integrated labour used up for its production and on the price of the corresponding intermediate commodity $k_i$ only. The price of each unit of vertically hyper-integrated productive capacity $k_i$, in its turn, depends on the cost of vertically hyper-integrated labour employed for its production — and, in the more complex formulation (see Garbellini 2010b, appendix A.4) also on its own price — only. In fact, inter-industry relations within vertically hyper-integrated sectors are reduced to the exchanges between each industry producing a final consumption commodity and the one producing the corresponding unit of productive capacity.

However, when all inter-industry relations are re-introduced into the picture, the price of each consumption commodity depends on the price of *all* the others, in the case of both final consumption commodities and units of vertically hyper-integrated productive capacity. Value creation thus depends on a very complex network of relations, reproduced to a sector-specific extent within *each* vertically hyper-integrated sector. In fact, all (basic commodities producing) industries enter all vertically hyper-integrated sectors.

This is also reflected in the analysis of labour productivities. In Pasinetti (1981), each rate of change of vertically hyper-integrated labour productivity $\varrho'_{i,t}$ $(i = 1, 2, \ldots, m)$ is the weighted average of changes in direct labour productivity in the industry producing final consumption commodity $i$, i.e. $\varrho_{i,t}$, and in the industry producing the corresponding intermediate commodity $k_i$, i.e. $\varrho_{k_i,t}$, the weights being the relative importance of the two industries in constituting the whole sector. But as soon as all inter-industry relations are concerned, and vertically hyper-integrated sectors are considered, besides having the vertically hyper-integrated component too — i.e. besides considering labour productivity in the production of additional productive capacity — the very changes in the (vertically hyper-integrated) labour productivity for the production of intermediate commodities $\varrho_{k_i,t}$ become weighted averages of *all* the $\varrho_{i,t}$s, since productive capacities made up not by one single commodity, but by *all* the (basic) commodities produced in the economic system as a whole. We can therefore think of many decompositions of vertically hyper-integrated labour productivities, to analyse the production of consumption commodities, of the different units of productive capacities, of their own production, and so on, according to the specific 'layer' of the productive structure, and of the structure of the single vertically hyper-integrated sectors, we want to analyse.

Differently from the case of the relative price system, the physical quantity side of the production process, both in a single period of time and through time, is not affected by the introduction of the more complete description of the technique in use in every points in time. The fact of using productive capacities as units of measurement for intermediate commodities prevents it from complicating the analysis, since it involves more complex changes in the *composition* of such productive capacities, which is a problem, as it has been stated in section 7, at page 110, that can be kept separated from that of the analysis of capital accumulation, precisely thanks to the adoption of such a unit of measurement.

Hence, the introduction of the complete inter-industry relations mainly affects the *value creation side* of the production process.

On the contrary, the dynamics of physical quantities — specifically, the dynamics of the 'equilibrium' quantities of the unit of productive capacity to be produced in each vertically hyper-integrated sector, and therefore capital accumulation — become much more complicated with the introduction of discrete, rather than continuous, time and thus of non-steady rates of change of demand

and labour requirements. In particular, the rate of change from time period $t$ to time period $t+1$ of the number of unit of productive capacity in capital stock available at the beginning of the production process is given by the sum of the rate of change of demand from $t-1$ to $t$ — and therefore on the evolution of such a stock through all the past time periods — and from $t$ to $t+1$.[15] This is a consequence of the fact that the very dynamics of capital accumulation given by *capital accumulation conditions* (4.14) involve the variation of demand for consumption commodities from the currently considered to the following time period, i.e. $c_{t+1}$. Though being quite an intuitive result, since capital accumulation's task is that of providing the economic system, in the *future* time periods, with an adequate stock of productive capacity, it is not possible to singling it out using continuous time — a choice that, as explained in Garbellini (2010c, section 3.4), is made by Pasinetti for a matter of analytical convenience; introducing non-steady rates of change of the main observable, exogenous variables would make the convenience of making such an assumption disappear, therefore making it more reasonable to use discrete time.

The introduction of more complex movements through time of demand and labour requirements, thus, mainly affects the *physical quantity side* of the production process.

A further final remark that I would like to make here concerns the reformulation, with respect to Pasinetti (1981), of what he calls the 'standard rate of growth of productivity', and therefore of the 'dynamic standard commodity'. As stated in section 9.3, the main property of such a composite commodity is that, when used as the *numéraire* of the price system, it makes the average price level constant through time. Its definition thus is the answer Pasinetti tries to give to Ricardo's problem of finding an 'invariable standard' of value. However, Pasinetti's (1981) does, in my opinion, suffer from two shortcomings.

The first shortcoming is that, in defining the 'standard rate of growth of productivity', Pasinetti (1981) does not take into account the role played by the rates of change of (average) per-capita demand on the composition of the basket of goods defining, in each time period, the real purchasing power of the average consumer, and therefore on the very definition of average price level.

The second one is that such definition is valid only *within the natural economic system*; as explained at the end of section 9.3, when the rate(s) of profit are not the 'natural' ones, and therefore prices are different from (vertically hyper-integrated) labour values, the logic adopted by Pasinetti (1981) in defining the 'standard rate of growth of productivity' does not work anymore. By reformulating the 'standard rate of growth of productivity', and thus the 'dynamic standard commodity', as in

[15] See equations (7.1) and (7.3).

expression (9.24), both this shortcomings, in fact closely connected to each other, have been overcome.

To conclude, what emerges from this attempt at extending to 'production in the long run' the generalisation of Pasinetti's (1981) framework developed in Garbellini (2010b) is that the analysis of capital accumulation, and in general of the structural dynamics of the physical quantity side of the economic system, is not complicated by the introduction of the more complete description of the technique in use; Pasinetti's contention — i.e. that the adoption of vertically hyper-integrated sectors as the object of the analysis, and of the units of vertically hyper-integrated productive capacity as the units of measurement for intermediate commodities, does make it possible to overcome the difficulties in the analysis of capital accumulation entailed by the presence of growth and technical progress — is fully confirmed. The analysis can however be enriched, and its degree of realism improved, by adopting discrete time and allowing for non-steady rates of growth of per-capita (average) demand for consumption goods and of labour productivities.

On the other hand, considering the network of inter-industry relations in all its complexity allows us to make the whole framework suitable for empirical applications and institutional analysis; that is to say, it allows us to take full advantage of all the reflections, suggestions and intuition put forward by Pasinetti's (1981) book, and to catch up in the most fruitful way the intellectual heritage of Piero Sraffa, that today, fifty years after the publication of his masterpiece, is still awaiting to exploit all its potentialities.

# Vertical Hyper-Integration and the 'Natural' Economic System — A reply to criticisms

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**Abstract** Pasinetti's (1981) book — and, in general, Pasinetti's vertically (hyper-) integrated framework — has been the subject of many reviews, and of many criticisms. Some of these criticisms are actually due to some ambiguity, in Pasinetti's exposition — or to the fact that Pasinetti's (1981) framework has been developed starting from simplifying assumptions that, though being functional to the development of the main idea at the core of this approach to economic analysis, are quite irrealistic.

The aim of the present paper is that of replying to such criticisms by taking advantage of the conceptual *excursus* made in Garbellini & Wirkierman (2010b) through Pasinetti's (1981) work, and of the generalisation carried out — starting from the hints provided by Pasinetti (1988) — in Garbellini (2010b) and Garbellini (2010a). The task is that of arguing that Pasinetti's device of vertical hyper-integration is a powerful tool to study economic reality, and therefore that it is necessary to further develop it, in order to provide economic analysis with an alternative approach — rooted in the Classical-Sraffian tradition but overcoming its difficulties — and therefore able to deal with the most important characteristics of modern industrial societies: structural change, technical progress and economic growth.

**Keywords** Natural system, vertically integrated sectors, vertically hyper-integrated sectors, functional income distribution, natural rates of profit, natural prices.

**JEL classification** B51,O41

## 1 Introduction

Pasinetti's (1981) book — and, in general, Pasinetti's vertically (hyper-)integrated framework — has been the subject of many reviews, and of many criticisms. Some of these criticisms are actually due to some ambiguity, in Pasinetti's exposition — or to the fact that Pasinetti's (1981) framework has been developed starting from simplifying assumptions that, though being functional to the development of the main idea at the core of this approach to economic analysis, are quite irrealistic.

These criticisms can be grouped into two categories.

The criticisms in the first category can be overcome by simply clarifying some points that have not been grasped, or have been grasped only partially — that is to say, such criticisms originate from a failure in fully understanding Pasinetti's analytical framework.

The criticisms in the second category, on the other hand, are due to the failure in understanding the *aim* of Pasinetti's (1981) book, as opposed to that of Pasinetti's (1988) paper (which is likely much less known than Pasinetti (1973)): while the latter provides a first step to formulate the whole analytical framework in a really general way — with all inter-industry relations and with the more complex description of the technique in use — the former's aim is that of showing the mechanism at the basis of vertically hyper-integration. In order to do so, many simplifying assumptions have been introduced, to get rid of analytical complications, and focus attention on the peculiarities of this kind of approach.

In other words, Pasinetti's (1981) book is the first building block for the understanding of the whole vertically hyper-integrated approach — actually, as stressed in Garbellini & Wirkierman (2010b), it also is the intermediate step leading Pasinetti himself to the complete and explicit formulation of this approach. From an analytical point of view, it is built on a number of simplifying assumptions which make it unsuitable for an *immediate* implementation for empirical investigations.

The starting point towards the accomplishment of this latter task is Pasinetti (1988); from there on, the way is open and waits to be explored. Anyhow, it is in Pasinetti (1981) that the really deep and *novel* theoretical and conceptual implications are drawn, and the problems affecting economic analysis since the time of the Classics faced. Many hints are given on how to go straight to the accomplishment of the above-mentioned task, too.

It is therefore my contention that Pasinetti's work has to be seen as a unitary *corpus*,[1] providing *both* deep and thought *theoretical* insights *and* clear indications of the way to be followed for *empirical* investigations.

The aim of the present paper is therefore that of taking advantage of the conceptual *excursus* made in Garbellini & Wirkierman (2010b) through Pasinetti's (1981) work — in order to reply to the first category of criticisms — and of the

[1] As explained in detail in Garbellini & Wirkierman (2010b), all Pasinetti's work, starting from his doctoral dissertation, can be seen as a series of steps finally leading to Pasinetti (1988), i.e. to the explicit presentation of vertically hyper-integrated sectors as a tool for economic analysis.

generalisation carried out starting from the hints provided by Pasinetti (1988) in Garbellini (2010b) and Garbellini (2010a) — in order to reply to the second category of criticisms. The task is that of showing that Pasinetti's device of vertical hyper-integration is a powerful tool to study economic reality, and therefore that it is necessary to further develop it in order to provide economic analysis with an alternative approach — rooted in the Classical-Sraffian tradition but overcoming its difficulties — and therefore able to deal with the most important characteristics of modern industrial societies: structural change, technical progress and economic growth.

All the criticisms that will be considered in what follows come from two kinds of sources: review articles, in particular on Pasinetti (1981), and discussions emerged during conferences or meetings in which Pasinetti's work has been the object of discussion. In some cases, the discussions have been directly stimulated, or caused, by the presentation of papers by myself (in particular Garbellini & Wirkierman 2010b, Garbellini & Wirkierman 2010a).

# 2 *Structural Change and Economic Growth*

## 2.1 Normative analysis as opposed to positive analysis

We can summarise very effectively this first criticism to Pasinetti's (1981) framework by quoting a review article — actually a very critical one — written in 1982 by Harris:

> Pasinetti's 'structural dynamics' is constrained within the requirements of his especial conditions of equilibrium, albeit a *moving* equilibrium, and, oddly enough, equilibrium is itself conceived as a kind of 'natural' state. Here one runs up against a problematical feature of this analysis that needs to be pursued further.
>
> (Harris 1982, p. 29)

Harris — but he is not alone — had clearly read the book as an attempt to perform a *positive* analysis, that is to say to describe what actually happens in a specific, *capitalist*, economic system: "it is presumed that the analysis is applicable to real functioning capitalist economies" (Harris 1982, p. 40).

Probably, the misunderstanding partially flourishes from the often made association between Pasinetti's and Kaldor's names, due to the well-known contributions they gave to the Classical/Keynesian theory of income distribution, sometimes not very accurately collected under the collective name of 'Kaldor-Pasinetti' theory.[2] This assertion is also confirmed by the words of another commentator:

[2]See Kaldor (1955) and Pasinetti (1962b).

> Pasinetti (like Kaldor) goes further [...] by also postulating that there is full employment in the natural system. Output is then fixed by labour supply.
>
> (Taylor 1995, p. 699)

Kaldor's analysis, actually, was a positive one: he tried to provide a 'Keynesian' theory of income distribution, where profits were considered as a prior claim on the share of the net output of an economic system, and then arriving at the so called *Cambridge equation*, relating the rate of profit and the rate of growth of the economic system. Or, to state it using Pasinetti's own words:

> A second and separate problem concerns the interpretative value of the model. When Mr Kaldor presented his theory of income distribution, he pointed out that the interpretative value of the theory depends on the *Keynesian hypotheses* on which it is built. [...]
>
> But this is not the approach that I should like to take here. Whether we are or whether we are not prepared to accept the model in this behavioural sense, there are important practical implications which are valid in any case. I should look, therefore, at the previous analysis simply and more generally as a logical framework to answer interesting questions about what *ought* to happen if full employment is to be kept over time, more than as a *behavioural theory* expressing what actually happens.
>
> (Pasinetti 1974, pp. 118-119, emphases added)

As explained at length in Garbellini & Wirkierman (2010b, sections 3.5 and 5), Pasinetti's (1981) framework is intended to find out those *physical* requirements that, *if* met, guarantee full employment of the labour force, full expenditure of national income and full utilisation of (vertically hyper-integrated) productive capacity. The dynamic *capital accumulation conditions* are precisely those requirements that *must* be met for the amount of new investment to drive capital accumulation in line with the pace of the dynamics of final demand for consumption commodities. "They are true whatever individual behaviour may be; as a simple matter of *logical necessity*" (Pasinetti 1974, p. 119; emphasis added).

No automatism is implied: reaching the objective in one period does not guarantee to be again in a *situation* of equilibrium in the following one. Physical requirements for new investment must be met period after period, in a constant actively pursued search for the new equilibrium situation. "If full employment is to be maintained [once reached], *that* amount of investment *must* be undertaken" (Pasinetti 1974, p. 119). No reference to a *capitalist* economic system is made: the institutional set-up[3] is left outside the analysis. The way in which such conditions

[3] That is, the set of social relations of (re-)production. See Garbellini & Wirkierman (2010b, section 2).

can be met within different institutional frameworks is a subject to be discussed separately, in a different stage — the *institutional* one — of the analysis.

Therefore, equilibrium is *not* conceived as "a kind of 'natural' state". On the contrary. Such a "state" can be reached if, and only if, it is actively pursued as an agreed end of the existing institutions.

The 'natural economic system' is therefore defined as an ideal, *normative* dynamic system in which the set of physical requirements described above — the macroeconomic condition for full employment of the labour force and full expenditure of the national income and the sectoral capital accumulation conditions — are met; the 'natural' rates of profit single out that amount of investment that *must* be undertaken if we want such conditions to actually be met period after period.[4] The structure of such an 'ideal' system is continuously changing through time, due to technical progress, changes in individuals' real income, shifting of per capita demand for consumption commodities.

Once these fundamental characteristics and aims of Pasinetti's (1981) framework are fully grasped and understood, it is therefore hard to maintain that

> [...]the natural economic system is essentially a *golden-age equilibrium* of a very special kind. It is one in which all of the structural changes which the author believes it is important to analyse unfold in full view as time goes by. These are the changes which, if we are to accept the stylised facts, do happen in real life and sometimes with disastrous consequences. But in the Pasinetti-golden-age they happen without any disturbance, specifically in regards to the condition of full-employment. Whereas in the ordinary golden age nothing happens, at least as far as changes in the 'structure' are concerned, in this particular golden age all sorts of changes occur and still there is full employment. We could just as well call it, therefore, a *super-golden-age.*
>
> (Harris 1982, pp. 40-41)[5]

To conclude, equilibrium is not *imposed* by Pasinetti (1981); the requirements for its realisation *period after period* are singled out, in order to stress the physical new investments — i.e. capital accumulation — necessities of the economic system.

## 2.2 Over-determination of the equation systems

Closely connected to the criticism mentioned above in section 2.1, we may add that some commentators objected that the effective demand condition is a rank

[4]Incidentally, this means that, with constantly changing rates of growth of final demand for consumption commodities, the natural rates of profit are themselves continuously changing through time too.

[5]Clearly enough, when talking about *golden ages* Harris is referring to Joan Robinson. See Robinson (1958) and Robinson (1962).

condition over-determining the system by *imposing* full employment. An example of such a criticism has been put forward by Parrinello:

> [...] in an unpublished paper in 1967 to Pasinetti's multi-sectoral model (Pasinetti, 1965, 1993). In fact, this model imposes a rank condition that guarantees the persistence of full employment in the presence of technical progress, which is assumed to be exogenous but obeying that condition.
> (Parrinello 2004, p. 319)

As explained in detail both in Garbellini & Wirkierman (2010b) and in Garbellini (2010b), the effective demand condition is a *macroeconomic* condition that, *if* satisfied, guarantees full employment of the labour force and full expenditure of the national income.

From a strictly mathematical point of view, it is obtained by Pasinetti (1981) as a condition for getting non trivial solutions out of both the physical quantity and the relative price system. When the two systems are reformulated as eigenproblems (see Garbellini 2010b, sections 3 and 5), it is a condition for having a unitary eigenvalue, to which the solution vector we are looking for is associated.

At the end of the previous section we stressed that Pasinetti (1981) does not *assume* equilibrium, but looks for the conditions to achieve it. In the same way, he does not *impose* full employment — which is one requirement coming from Pasinetti's (1981) very definition of an equilibrium *situation* — but looks for the conditions that, *if* satisfied, imply a state of full employment in the system.

Therefore, what he does when formulating the quantity and price systems, is closing them with two expressions — one for each — describing a situation of full employment of the labour force and of full expenditure of national income. If such equations hold, the resulting systems describe a situation of *flow* equilibrium, and the corresponding solutions are 'equilibrium' solutions; their mathematical condition of existence provides us with a formal relation indicating the requirements that must be met, from an economic point of view, for such an equilibrium situation to be realised.

The criticism again involves the misunderstanding of the normative, rather than positive, nature of Pasinetti's (1981) framework. He is not *assuming* equilibrium in order to describe what actually happens in a concrete economic system. He is *describing* equilibrium in order to arrive, by means of formal logic alone, to the conditions that *must* be realised if an equilibrium situation is to be achieved.

Going back to the mathematical aspect of the problem, saying that the macroeconomic condition is a rank condition amounts to saying that the last equation of the quantity and price systems, respectively, are over-determining the corresponding system. As shown elsewhere (see Garbellini & Wirkierman 2010b, pp. 6-7), these last equations can be modified in order to allow for the non-realisation of full employment of the labour force — within the quantity system — and full expenditure of the national income — within the price system. When such a procedure is adopted, the expression for the macroeconomic condition changes from:

$$\sum_i a_{in}(t)a_{ni}(t) + \sum_i T_i^{-1} a_{in}(t)a_{nk_i}(t) + \sum_i a_{k_i n}(t)a_{nk_i}(t) = 1 \tag{2.1}$$

to

$$\sum_i a_{in}(t)a_{ni}(t) + \sum_i T_i^{-1} a_{in}(t)a_{nk_i}(t) + \sum_i a_{k_i n}(t)a_{nk_i}(t) = \alpha \gtreqless 1 \tag{2.2}$$

Expression (2.1) is a *macroeconomic condition* for flow equilibrium, i.e. a 'normative' relation; expression (2.2) is, by contrast, obtained from two equation systems written down to *describe* a contingent situation — in which flow equilibrium is not realised — and therefore is a 'positive' relation describing what happens at the 'macroeconomic', but it would be better to say aggregate, level. That is to say, by adopting such a procedure we exit the *foundational* stage of the analysis, entering the *institutional* one, which was not Pasinetti's (1981) aim.

Incidentally, Parrinello (2004) also adds that

> [...] still maintain[s his] previous critical assessment of Pasinetti's model from a theoretical point of view: normal prices are not associated with persistent full employment. However, we cannot charge with inconsistency a model because its system of equations becomes over-determined in the absence of a constraint that the model builder explicitly imposes on its parameters.
>
> (Parrinello 2004, footnote 13, p. 321)

In this respect, it is worth stressing that Pasinetti's notion of 'natural prices' is different from that of 'normal prices' coming from the so called 'surplus approach'. It is therefore out of place criticising the former for not being consistent with the characteristics of the latter.

## 2.3 Pre-institutional theory of income distribution

A further, quite spread, criticism can also be very effectively summarised by using the reviewer's own words:

> [Pasinetti's] attempt to develop his production system independently of institutional features runs into difficulties as soon as he deals with prices and distribution. 'Profits' is a term with meaning only in a capitalist society. His 'natural' rates of profit, that in equilibrium model provide the finance required in each sector to maintain the sectoral equilibrium rates of growth, would be 'natural' rates of tax in a socialist economy. The only category of income in the latter would be wages.
>
> (Asimakopulos 1982, p. 1566)

The argument concerning the characteristic of Pasinetti's (1981) pre-institutional theory of income distribution has been carried out in detail in Garbellini & Wirkierman (2010b, section 4.1) and it is therefore not necessary to go deep into this issue. Suffice here to recall that "at a pre-institutional stage of the analysis, a theory of the rate of profit is *not* a theory of income distribution among income recipients, [...] because the very definition of the categories among which the purchasing power generated in the process of production is to be distributed essentially depends on the social relations of production of a particular institutional set-up" (Garbellini & Wirkierman 2010b, p. 21). Anyway, "prices of production provide for the purchasing power both to self-replace and expand productive capacity and to consume those commodities not re-entering the circular flow. Consider that profits and wages just establish the amount of purchasing power that must be channeled to demand for means of production to expand productive capacity and to demand for final consumption commodities, respectively" (Garbellini & Wirkierman 2010b, p. 22).

Moreover, it is worth spending a few lines on a further issue emerging from Asimakopulos's (1982) critique quoted above, i.e. that "[p]rofits' is a term with meaning only in a capitalist society". I would say that 'profits' is a term which would not make sense if referring to a *pre-industrial* society; it was born together with the *capitalistic*[6], i.e. industrial, mode of production, at the time of the Industrial Revolution. Of course, the first industrial societies were *capitalist* ones; in fact, we have to acknowledge that *capitalistic* economic systems with different institutional set-ups did in their turn emerge as a reaction to the capitalist social relations of production.

Therefore, while we could not use the term 'profits' when dealing with a pre-industrial economic system — we could not talk about capital accumulation actually, and therefore the necessity of using such a term would not even arise — we can perfectly think of profits within any kind of *capitalistic* economic system. The necessary and sufficient condition for the term 'profits' to make sense is the existence of a process of *physical* capital accumulation, whatever the social relations of production within which it takes place — and therefore whoever appropriates profits themselves, be them the capitalists, a central authority, or someone else.

A *uniform* rate of profit is a characteristic of capitalist societies; the natural ones, being different from sector to sector — and once the description of the technique in use is generalised, also leading to different prices for the very same commodities according to the growing subsystem they belong to — could not be realised within this institutional framework. But 'profits' are simply a component of the production prices, exceeding the labour costs and the costs associated to the

[6]For an explanation of the difference between the term 'capitalist' and the term 'capitalistic', see Garbellini & Wirkierman (2010b, section 2).

reproduction of used-up intermediate commodities; such a component is computed in proportion to the stock of accumulated capital, evaluated at current prices.

## 2.4 Labour as the only non-reproducible factor of production

The last criticism to Pasinetti's (1981) *Structural Change and Economic Growth* that I want to consider here is that "Dr. Pasinetti assumes that labour is the only scarce factor of production" (Champernowne 1964, p. 660). More precisely,

> [t]he analysis ignores the role of natural resources. This approach seems reasonable as a first step, because it properly assigns conceptual priority to reproducible commodities. But it would seem necessary to grant, even at this level, as long as technical progress is the main focus of analysis, that the rate and direction of such technical progress may be significantly conditioned by the economic stimulus that comes from the dynamics of natural-resource utilisation. Consumption patterns may also be similarly influenced. [...] There are significant aspects of the process of uneven development and disproportionality of growth that are not captured in this analysis.
>
> (Harris 1982, p. 39)

This kind of criticism is not an isolated one, especially nowadays that the issues of 'sustainable development' and management of exhaustible natural resources have become very fashionable in economic analysis, both in the mainstream and among heterodox economists — input-output analysis, particularly, seems pursuing more and more this research line, being used for environmental applications such as 'industrial ecology'.

The fact that natural resources are important is not denied, of course, as Pasinetti himself stated in his Doctoral Thesis:

> My impression is that the problems of scarcity are theoretically very exciting; and yet in practice have not had the importance which our theories have tended to give them.
>
> The bulk of contemporary economic theory has started from the investigation of the optimum allocation of scarce resources in an absolutely stationary world; and has *then* tried to extend the same concepts to a growing economic system. I am proposing a theoretical model which starts from the opposite end; namely from an economic system in which there is no scarcity but there is learning and thus economic growth. Later on — I am hoping — it may well turn out to be easier to introduce scarce resources into a model for learning and growth than it has been so far to introduce learning and growth into a model of scarce resources.
>
> (Pasinetti 1965, p. 695)

However, there is a difference between using a theoretical framework for studying a particular, concrete, *institutional* problem — e.g. accounting for greenhouse gas emissions associated to the production of a particular final commodity[7] — and introducing a particular, concrete, *institutional* problem into the *foundational* basis of the framework itself.

Scarce resources cannot be the basis of the production paradigm, as Pasinetti calls it. Labour is considered as the only *primary* factor of production because without labour, without *human effort*, no commodities can be produced:

> Nothing in the present theoretical scheme has any economic relevance — i.e. value — other than in relation to the activity and wants of the members of the community. What nature offers is a datum — it is taken for granted. Any commodity, by itself, has no personality: it has no right or claim. Of course, commodities do physically produce other commodities — machines produce machines, animals reproduce animals — but this 'physical' productivity must be correctly interpreted. Commodities cannot appropriate the commodities that come out of them. Only Man can. The physical productivity of commodities simply is a part of their technical or biological properties, which for Man is a datum. What becomes relevant, for economic purposes, which means for the process of pricing, is only the amount of human activity which is required, whether directly or indirectly [or hyper-indirectly], to make a technological or biological process work.
>
> (Pasinetti 1981, p. 131)

On the contrary, all other non-produced factors of production can be substituted with different ones — e.g. oil, by finding alternative fuels — or exploited in a more efficient way to overcome the problem of its scarcity — e.g. land, thanks to technical progress. The argument can once again be better presented by borrowing someone else's words; in this case, Sraffa's:

> But how are we going to replace these natural things? There are 3 cases: a) they can be reproduced by labour (land properties, with manures and so on; b) they can be substituted by labour (coal by hydroelectric plant: or by spending in research and discovery of new source and new methods of economising) c) they cannot be either reproduced nor substituted and in this case they cannot find a place in a theory of *continuous* production and consumption: they are dynamical facts, that is a stock that is being gradually exhausted and cannot be renewed, and must ultimately lead to destruction of the society. But this case does not satisfy our conditions of a society that just manages to keep continuously alive.
>
> Sraffa Papers D3/12/42: 33, from Kurz & Salvadori (2002, p. 408)

[7] This is one of the above mentioned environmental applications of input-output analysis, called 'carbon footprint'.

Up to the Mercantilist era, "the wealth of a nation was identified with the wealth of its king" (Pasinetti 1977, p. 2). But with the breakthrough of the Industrial Revolution, there has been a change of emphasis — started by the Physiocrats and then taken up by the Classical economists — from the problems concerning the scarcity of natural resources to those concerning *produced commodities*: "it was no longer, or not so much, the distinction between wealth as a stock and wealth as a flow that was seen as important, but rather the contrast between *produced* wealth (whether as an annual flow or as an accumulation of means of production) and exogenously given natural resources" (Pasinetti 1977, p. 3).

This theoretical shift reflects, according to Pasinetti (see for example Pasinetti 1965, Pasinetti 2007) the historical shift from the phase of trade — i.e. the pre-industrial era, "perceived even as early as at the turn of the first millennium" (Pasinetti 1965, p. 573) — and the phase of industry; the former is based on *exchange*, "by a better spatial allocation of existing resources and products" (Pasinetti 1965, p. 573); the latter on *production*, i.e. "a process of augmenting wealth through a material increase in the quantity and number of products, to be reached by the practical application of the advantages of science, division and specialisation of labour, better organisation, invention and utilization of new resources of energy and new materials" (Pasinetti 1965, p. 573).

To show how this historical shift caused the theoretical one, Pasinetti stresses that during the phase of trade — an intrinsically *static* concept — the economists' concern was "the problem of how to reach the best allocation of given resources" (Pasinetti 1965, p. 574); on the contrary, the phase of industry — an intrinsically *dynamic* concept — brought about a whole series of new challenges, connected to the necessity of re-organising society and finding new and better methods of production: "[t]he economist is faced here no longer with a problem of rationality, but with a *process of learning*" (Pasinetti 1965, p. 575).

Pasinetti's conclusion is therefore that

> these are two distinct series of *problems*. A particularly important difference between the two, for theoretical analysis, is that they acquire an opposite practical relevance in relation to time, the former being relevant (in the short run) just when the latter is practically irrelevant and the latter becoming relevant just when (in the long run) the former becomes irrelevant.
>
> (Pasinetti 1965, p. 575; emphasis added)

I did recall here — even if very briefly (for details, see Pasinetti 1965, pp. 572-575) — this historico-theoretical *excursus* because it gave rise to a further criticism, concerning the whole argument but particularly the above-quoted conclusion:

> The classical surplus theories are characterized by some authors as being concentrated on reproducible commodities, and hence "production", as opposed to the concentration on commodities of the scarcity type and hence on

> "exchange" which would be the hallmark of the dominant marginalist theories. Accordingly the two kinds of theory would deal with two distinct series of problems, with an opposite practical relevance in relation to time, the classical theory becoming relevant just when (in the long run) the marginalist theory becomes irrelevant [(cfr. e.g. Pasinetti 1965, pp. 573-575)]. Whereas it aptly describes some differences between the two approaches, this distinction seems not to go to the roots of the difference, which lies in the way in which both "production" and "exchange" are treated in each approach.
>
> (Garegnani 1984, p. 298, footnote 15)

This is a distorted interpretation of Pasinetti's passage. In that passage, Pasinetti is referring to two different set of *problems*, not about two different theoretical paradigms. The contention is that, when dealing with problems related to optimal allocation of given resources, one deals with an essentially *static* problem, which can be faced as the rational choice of how to allocate an already existing endowment of 'wealth' in order to reach a certain objective. The relevance of these problems is therefore confined to the short run.

On the contrary, the analysis of problems related to industrial production is intrinsically dynamic: there are changes, induced by technical progress, that result from slow but persistent processes that can be perceived only in the long run, i.e. in a *dynamic* context; the temporal dimension cannot be disregarded, but has to be considered as the standpoint of the analysis. The Physiocrats first, and the Classical economists afterwards, had perfectly understood the importance of developing a *production paradigm*; Marx, in particular, was precisely working in such a direction.

The marginalist revolution — or better, counter-revolution — happened to take place precisely in the middle of these great historical, social struggles that brought modern industrial economic systems into existence. Nonetheless, marginal theory has been developed only with reference to the rational problem of optimal allocation of scarce resources, which quite obviously sounds as a contradiction: a new theoretical paradigm emerging after an unprecedented social and historical change should drive attention to the *new* problems, not bring it back to the *old* ones.

This is to say that Pasinetti asserts that the *focus* of marginal theory is on scarce resources, and that, as a consequence, its *method* of analysis allows to deal only with static problems. But he did not say that the marginalist way of doing so is the correct one to deal with the issue of exchange. He did not say that marginalist theory is relevant in the short run while Classical theory is relevant in the long run. He did say that the problems on which Classical theory is focused are relevant in a *dynamic* framework; while the problems marginal theory deals with — though not in the correct way — are relevant in a *static* framework.

# 3 Vertical hyper-integration and growing sub-systems

## 3.1 Pasinetti and Sraffa

Some criticisms raised against Pasinetti's framework concern the fact that the use of *vertical (hyper)-integration* adds nothing to what can be already concluded by using Sraffa's sub-systems or a standard multi-sectoral (industry-level) model. To a greater extent, it is sometimes asserted that vertical (hyper-)integration disregards inter-industry relations, whose description is one of the major achievements of multi-sectoral analysis. In sum, Pasinetti's framework does not have very much to add to economic theory, and comes to be an elegant but not very useful elaboration.

We will concentrate on the issue of inter-industry relations, and on the consideration of the circular flow, later on, in section 3.3. Let us therefore start by considering the first part of such criticism, analysing the relation between Sraffa's sub-systems and Pasinetti's vertically integrated and hyper-integrated sectors.

As Pasinetti (1973) points out, the notion of vertical integration is very widely used in economic analysis — even if often without full awareness — and not only within non-neoclassical frameworks, but in a multiplicity of contexts of very different nature:

> The notion of vertical integration is implicit in all discussions on the theory of value of the Classical economists. The same thing can be said of the marginalist economists. When, for example, Léon Walras adopted the device of eliminating intermediate commodities from his analysis of production, he was making use of the logical process of vertical integration. Keynesian macroeconomic analysis is also generally carried out in terms of vertically integrated magnitudes (net national income, net savings, new investments, consumption, and so on). Very rarely, however, is the logical process of vertical integration explicitly discussed. Generally it is simply taken for granted.
>
> (Pasinetti 1973, p. 1)

Clearly, identifying the core — and the originality — of Pasinetti's contribution with the device of vertical integration is not simply reductive, but inadequate, as such a device has been used by a great number of economists, in a great number of different periods, situations, and within different theoretical frameworks.

I have already discussed at length, elsewhere, the difference between vertically integrated and hyper-integrated sectors.[8] Suffice here to recall some basic points.

First of all, Pasinetti's (1973) vertically integrated sectors represent an attempt at analytically formulating Sraffa's (1960) *sub-systems*, as it should be clear by reading Sraffa's own words:

[8] See Garbellini (2010b, section 4).

> Consider a system of industries (each producing a different commodity) which is in a self-replacing state.
>
> The commodities forming the gross product [...] can be unambiguously distinguished as those which go to replace the means of production and those which together form the *net product of the system*.
>
> Such a system can be subdivided into *as many parts as there are commodities in its net product*, in such a way that *each part forms a smaller self-replacing system* the net product of which consists of only one kind of commodity. These parts we shall call 'sub-systems'.
>
> [...] Although only a fraction of the labour of a sub-system is employed in the industry which directly produces the commodity forming the net product, yet, since all other industries merely provide replacements for the means of production used up, *the whole of the labour employed can be regarded as directly or indirectly going to produce that commodity.*
>
> (Sraffa 1960, p. 89, emphases added.)

Actually, *at a single point in time*, sub-systems and vertically integrated sectors are the same thing; or better, the latter are a compact way of describing the former. Both sub-systems and vertically integrated sectors are a way of re-classifying the production processes that take place in the economic system — alternative to the more usual and directly observable one based on industries — aimed at identifying and isolating all the direct and indirect processes that allow the production of the net output, i.e. *final demand*.

This essential coincidence, within a single period of time and in a *static* framework, of Sraffa's sub-systems and Pasinetti's vertically integrated sectors could lead to draw the conclusion that the latter has nothing to add to what the former has already said. But Pasinetti went further.

While Sraffa, as he explicitly said, limited his analysis to "taking a 'photograph' of an economic system, as this actually can be observed at a certain point of time" (Pasinetti 2007, pp. 189-190),[9] Pasinetti tries to overcome this limitation, analysing the *dynamics* of economic systems.

In order to do so, he redefines the notion of net output, in order to be able to treat *extended reproduction* avoiding the breaking up of the circular flow caused by the introduction of growth into the picture (See Garbellini 2010b, section 4.2). In particular, he separates that part of the net output that does not re-enter the circular flow, i.e. consumption commodities — from the one which does re-enter it *in the following period*, as additional productive capacity: new investments. Therefore, even if we are still in front of a way of re-partitioning the productive activities taking place in the economic system as a whole, the way in which such re-partitioning is effected is entirely different. It is a full generalisation of Sraffa's idea.

[9]Pasinetti is citing the *Sraffa papers*, C294/2.

More specifically, the gross product of vertically integrated sector $i$ is given by two components. The first one is a quantity $y_i = x_i + j_i$ of the *homogeneous* commodity $i$, sold at the end of the production period either to be consumed consumed ($x_i$) or be part of the *whole* economic system's new investments ($j_i$), i.e. to become meas of production in the following period(s). The second one is the set of *heterogeneous* commodities (re-)produced as the used up — both directly and indirectly — means of production. At the end of the production period, in order for the economic system as a whole to be provided with an increased productive capacity for the following one, each sector has to buy new investment goods from, and to sell a part of its net output to, the others. Hence, when we consider growth, the sub-systems are no more in a "self-replacing state".

On the contrary, the gross product of vertically hyper-integrated sector $i$ is made up by two components, but defined in an entirely different way. The first part is a quantity $x_i$ of the homogeneous commodity $i$ which is produced in order to be *consumed*. The second part consists of heterogeneous commodity produced in order to become means of production. They include *the whole set* of new investments commodities that are necessary to expand the sector's productive capacity — in line with the evolution of final demand for the corresponding consumption commodity — as well as that set of intermediate commodities that have to replace those used up during the production process. In this way, thus, each vertically hyper-integrated sector produces all the new productive capacity it needs: it does not need to buy part of their net output from, and sell part of its net output to, the others. The "self-replacing state" is recovered.

By going into dynamics, Pasinetti can analyse changes in the structure of physical quantities of the economy, instead of considering "[n]o changes in output and [...] no changes in the proportions in which different means of production are used by an industry" (Sraffa 1960, p. v), and hence overcome the second great criticism which has been raised against Sraffa's system, i.e. that of being only a 'half-system'. Vertical hyper-integration is the tool allowing to put together Leontief's concerns with the quantity, physical side of the production re-process and Sraffa's concern with the price, value side.

## 3.2 Fixed coefficients and exogenous technical progress

After having considered the relation between Pasinetti and Sraffa, it is worth devoting some time to stressing the analogies between Pasinetti and Leontief. This will open up the way to clarify a methodological characteristic of the whole framework developed by Pasinetti which has not been grasped in its full relevance, and therefore has given rise to a series of criticisms.

Pasinetti himself points out

> the similarity of approach, *from an empirical point of view*, of the previous dynamic (vertically [hyper-]integrated) analysis and the static input-output analysis. Both of them share the characteristic of being built on coefficients which are intended to represent *actual outcomes* and which can therefore [...] be given an *empirical content*, simply by recording the actual performance of an economic system. [...] The coefficients that appear both in the input-output analysis and in the present (vertically [hyper-]integrated) analysis must, therefore, be interpreted as representing those physical quantities which can actually be *observed*.
>
> (Pasinetti 1981, pp. 109-110; emphases added)

This excerpt is of fundamental importance for the understanding of Pasinetti's methodological approach. He departs from the very same statistical conception as Leontief. The coefficients appearing in the whole analysis are precisely, *period after period*, those magnitudes that can actually be observed and *measured*.

After acknowledging such analogy, one can therefore be tempted to criticise Pasinetti's framework with the same arguments used to criticise Leontief, by saying that he takes fixed coefficients, and makes the implicit assumption of constant returns to scale, because dealing with changing coefficients according of the scale of output would not be possible, or would be too difficult.

But this is not what Pasinetti does; he did not do so in 1981, and he did not do so in 1988.

As explained at length both in Garbellini & Wirkierman (2010b, sections 3.2 and 3.5) and in Garbellini (2010b, section 5), Pasinetti uses a particular unit of measurement for intermediate commodities, i.e. the units of vertically hyper-integrated productive capacity. In this way, it is possible to deal with capital accumulation by simply studying the dynamics of the stock of units of productive capacity, leaving aside the issue of their changing physical composition. The two problems are therefore separated so that each one can be analysed independently of the other:

> the notion of a physical unit of productive capacity, by being defined with reference to the commodity that is produced, continues to make sense, as a physical unit, whatever complications technical change may cause to its composition in terms of ordinary commodities. (Pasinetti 1973, p. 24)

Therefore, the fact that matrix **A** is continuously changing through time is not disregarded. In each period, the specific matrix considered is the one that can be obtained from national accounts.[10]

[10] In the analytical formulation of the framework, both matrix **A** and all the derived matrices (**H**, **M**, etc) are not dated so as not to make notation and calculations too complicated. But this does not entail any implicit assumption on the dynamic behaviour of inter-industry coefficients.

As to returns to scale, Pasinetti makes no specific assumption about them. The argument goes along the same line as above; it may well be that, were the gross quantities produced different, the coefficients would not be the same. In other words, it is not maintained that the coefficients we observe would be the same whatever the scale of production. Coefficient $a_{ij,t}$ does not represent the amount of commodity $i$ which, at time $t$, is necessary for the production of one unit of commodity $j$; it is the quantity of commodity $i$ that *has actually been used*, in period $t$, for producing each unit of the total quantity of commodity $j$ that has actually been produced. This is a crucial difference. We do not care about what could have happened in a different situation. We record what has actually happened, and measure the corresponding relevant magnitudes.

This argument brings about a further criticism concerning Pasinetti's (1981) book specifically, but can be better replied by also considering the more general formulation provided by Pasinetti (1988) and attempted in Garbellini (2010b) and Garbellini (2010a). Such a criticism concerns the 'feasibility' of the $\varrho'$s, i.e. of the rates of growth of labour productivity at the level of each vertically hyper-integrated sector:

> I am not entirely convinced that it is legitimate to express technical progress generally in terms of reductions of the inputs to these integrated sectors [...]. For technical change takes place at the industry level so that the rates of productivity growth in the different integrated sectors can not be thought of as being independent of each other. Moreover, it is easy to see that rates of productivity growth which are arbitrarily assumed at the level of integrated industries do not necessarily correspond to feasible (positive) rates of productivity growth at the level of 'ordinary' industries.
>
> (Schefold 1982, p. 549)

Besides the fact that, unfortunately, the rates of productivity growth *can* also be negative, Schefold is absolutely right in saying that "the rates of productivity growth in the different integrated sectors can not be thought of as being independent of each other". But in Pasinetti's framework, such rates are not "arbitrarily assumed at the level of integrated industries"; in the same way as all other *derived magnitudes*, they are *computed* from the *actual* rates of change of labour requirements at the industry level. Therefore, it is not necessary to ask ourselves about the feasibility of such rates. In Pasinetti's (1981) simplified framework, of course, the rate of productivity growth in the production of the consumption commodities ($\varrho_i$, $i = 1, 2, \ldots, m$) and of the 'capital goods' ($\varrho_{k_i}$, $i = 1, 2, \ldots, m$) are industry-level ones, since the technique is such that each vertical hyper-integrated sector is made up by to industries, one producing the consumption commodity and the other producing the corresponding intermediate commodity. Only the rate of change of productivity at the level of the sector as whole ($\varrho'_i$, $i = 1, 2, \ldots, m$) is a derived magnitude, being a weighted average of $\varrho_i$ and $\varrho_{k_i}$.[11] But as soon as we introduce the most general description of the technique, it is easy to see how $\varrho_i'$ is the weighted average of the rate of change of labour requirements in *all* the industries constituting the economic system as a whole (see Garbellini 2010a, section 5).

### 3.3 Vertical (hyper-)integration: circular flow and empirical relevance

Closely connected to the topic discussed above, in section 3.2, we have a further criticism concerning vertical (hyper-)integration and its connection to empirical facts:

> For the analysis of structural change [...] the relevant question is: does one lose useful information with this vertical integration manoeuvre?
>
> Unfortunately, the answer is yes. (Taylor 1995, p. 700)

I completely disagree with Taylor's (1995) conclusion. Also in this case, it is not an isolated opinion; it is not uncommon to hear reviewers objecting that with vertical (hyper-)integration the circular flow is lost — while using Sraffa's subsystems it is preserved.

Both vertical integration and hyper-integration are ways of repartitioning economic activities in a specific way: according to the single commodities composing the whole net product (i.e. consumption and new investment commodities) in the former case, according to the single consumption commodities in the latter. In both cases, such a re-partition is effected through a *linear transformation*, that can be easily reverted and thus preserving, in both directions, all the original information, since "once we possess the inverse matrix, all relations between the two approaches at a given point of time take the form of one-to-one correspondences" (Pasinetti 1981, p. 115).

Inter-industry relations, therefore, are not disregarded. On the contrary: they are still considered in all their importance.[12] Not only: with respect to traditional inter-industry analysis, they are considered in a more complex way, as not only *direct*, but also *indirect* — and in Pasinetti (1988) also *hyper-indirect* — relations are taken into account.

[11] See Pasinetti (1981, p. 103).

[12] The vector of *vertically integrated productive capacity* "contains the *series of heterogeneous commodities* that are directly and indirectly required in the whole economic system to obtain one physical unit of commodity $i$ as a final good". (Pasinetti 1973, p.5; emphasis added). Therefore they take into account the fact that (part of) the output of an industry is used by another industry as an input, and *vice versa*.

In the specific case of vertical integration — that, incidentally, is a way of formalising Sraffa's subsystems; if it is maintained that these latter preserve the circular flow, it cannot be maintained that vertically integrated sectors do not — the linear transformation is effected through the Leontief inverse matrix.

For vertical hyper-integration, the procedure, from an algebraic point of view, is precisely the same, with the only difference that the matrix we use for the linear transformation is not $(\mathbf{I}-\mathbf{A})^{-1}$ but $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ (with $i = 1, 2, \ldots, m$; see Garbellini 2010b, section 4.2).

> The [...] inverse matrix appears, therefore, as the linear operator which may be applied to an inter-industry classification of labour and capital goods, in order to reclassify them according to the new type of (vertically [hyper-]integrated) sectors.
>
> In this way, each vertically [hyper-]integrated sector is reduced to one flow-input of labour and one stock-quantity of capital goods; or, more specifically, to one vertically [hyper-]integrated labour coefficient and to one vertically [hyper-]integrated unit of productive capacity. [...] Formally, the new coefficients are, therefore, *derived* concepts (derived from the consolidation of inter-industry coefficients) but they have a deeper economic meaning and possess [...] much more favourable characteristics for dynamic analysis.
>
> (Pasinetti 1981, pp. 113-114)

The emphasis put by Pasinetti on the fact that vertically (hyper-)integrated coefficients are derived magnitudes with "a deeper economic meaning" brings about another remark made by Schefold in his review of Pasinetti's (1981) book:

> This does not mean that the concept of vertically integrated sectors is meaningless — on the contrary, it is very helpful —, but it illustrates the point that we have yet to examine the interdependence between different rates of productivity growth in integrated sectors and that the input/output structure retains its factual and conceptual priority over the derived concept of integration.
>
> (Schefold 1982, p. 549)

In this respect, it is worth stressing that Leontief's input-output model can be considered as the *static counterpart* of the vertically hyper-integrated framework. This means that there is no *logical priority* of one of them over the other: they are instruments to be applied to two *different, complementary*, problems:

> Over time, and as the conditions of production and of consumption change (owing to technical progress, economies and diseconomies of scale, etc.) the inter-industry relations break down and become different from one moment to the next, so that a particular input-output table is needed for

> each stage in the evolution of the economy under consideration. These tables can be compared, [...] but they cannot be analytically linked to one another [...]. The continuity in time is kept, on the other hand, at the vertically [hyper-]integrated level, where the relations which can be set up possess [...] a higher degree of *autonomy.* This means that the permanence of these relations in time is independent of technical change. In this context, the vertically [hyper-]integrated technical coefficients acquire a meaning of their own, *independent of the origin of the single parts which compose them.* The movements of these coefficients through time, and the various consequences thereof, can be investigated and followed as such. When more information is needed about the industrial structure at a particular point of time, the vertically integrated coefficients can be split and analysed into inter-industry coefficients particular to that point in time.
>
> In this way, static input-output analysis and dynamic vertically [hyper-]integrated analysis appear as *mutually complementary and completing each other.* Inter-industry relations, referring to any particular point of time, represent a cross-section of the vertically [hyper-]integrated magnitudes, whose movements through time express the structural dynamics of the economic system.
>
> (Pasinetti 1981, p. 117; second and third emphases added)

As to the *factual* priority of input-output relations, it would clearly not make sense to directly *collect* data on vertically hyper-integrated sectors instead of on industries, the latter being immediately observable and therefore easier to be recorded; whether to use a traditional input-output approach or the vertically hyper-integrated one, once again, depends on the kind of problems that we want to investigate.

Very closely connected to what we have been saying in the first part of the present section — i.e. how to fit actual data into a vertically hyper-integrated model, and the relation between Pasinetti's and standard input-output analysis — there comes a further criticism, concerning the empirical relevance of vertically hyper-integrated analysis. It is maintained that vertically integrated and vertically hyper-integrated analyses have no empirical relevance, especially when dealing with *technical change* and *productivity measures*, because any conclusion drawn could be equally drawn by using the standard input-output model, the only difference being that the latter would have an immediately clear economic meaning, while a meaningful economic explanation of the former could hardly be given.

Having already stressed the complementarity of vertically (hyper-)integrated and traditional input-output analysis, a first hint at what the empirical relevance of the former is can be given by providing some examples of its application in the literature.

### 3.3.1 Changes in labour productivity

A very interesting example of the use of vertical (hyper-)integration in empirical analysis concerns the construction of *productivity measures* alternative to the traditional measures of *multifactor productivity* obtained from a neoclassical aggregate production function.

The literature concerning productivity measures and their empirical evaluation is very rich, starting from Solow's (1957) well-known paper and the one side, and therefore from Pasinetti's (1959) critique and further exchanges[13] on the other side:

> There have been some attempts by economists to complete [evaluations of technical change] and to introduce capital into the picture, by making use of theoretical notions as the production function, but these attempts — in the writer's opinion — have neglected an important characteristic of capital — that it is *reproducible* and that its process of production is also subject to technical change. It is my purpose in this paper to go into these problems. I shall try to give a short economic interpretation of technical change and suggest a procedure for evaluating it, with respect to all factors of production.
> (Pasinetti 1959, p. 270; emphasis added)

This excerpt stresses an issue which is very important when dealing with changes in productivity: all the, reproducible, intermediate means of production are themselves subject to technical progress. Therefore, when measuring productivity changes, or changes in capital intensity, "the changes which occurred in the production of physical capital itself, i.e. the changes in productivity in capital goods industries" must be "explicitly taken into account" (Pasinetti 1959, p. 274).

Pasinetti's (1959) paper opened up a line of research based on empirical applications trying to compute changes in labour productivity in vertically integrated terms, explicitly acknowledging for the role of vertically integrated sectors in taking into account technical progress not only in the very production of each final commodity, but also in the production of all the intermediate commodities used up during the production process itself. The contention is that the phenomenon of technical change, and its consequences on the economic system as a whole, cannot be adequately understood but by considering *all* its effects on the production process; not only direct, but also *indirect* ones.

Without going into details here, let me mention some works which adopt a standpoint connected with vertically integrated analysis: Gossling & Dovring (1966), Gupta & Steedman (1971), Gossling (1972), Rampa (1981), Rampa &

[13]See Pasinetti (1998) and Solow (1998).

Rampa (1982), Ochoa (1986), Buccellato (1990), Rampa & Rampa (1990), Elmslie & Milberg (1996), Dietzenbacher, Hoen & Los (2000), De Juan & Febrero (2000), Fredholm & Zambelli (2009). See also Garbellini & Wirkierman (2010a).

### 3.3.2 Analysis of tertiarisation

Another field of application of vertically integrated analysis has been the analysis of the so-called process of tertiarisation in advanced industrial economies. The great majority of applied papers studying this topic have their starting point in the work by Siniscalco (1982) and Momigliano & Siniscalco (1986).

The topic of tertiarisation started to gain popularity in Italy between the end of the '70s and the beginning of the '80s, when data suggested that the manufacturing industries were losing importance — in terms of produced output and employment — with respect to the services industries.

In the meanwhile, however, almost all great firms were bringing about a radical change in their productive structure, i.e. an outsourcing of all those activities other than the core one, that were previously *vertically integrated* and therefore carried out within the firm itself.

The coexistence of these two phenomena brought the authors to the conclusion that there was the possibility that at least part of the increased relative importance of services with respect to manufacture could be due to this process of outsourcing. In the most extreme of all hypotheses, the growth of tertiary activities could even be the result of an *increase*, rather than a decrease, in industrial activity, therefore requiring a greater and greater amount of — externalised — services. But even in the smoothest case, industry-level data could be misleading, producing an over-estimation of the phenomenon.

Performing the analysis in terms of vertically integrated sectors — or sub-systems — could allow the authors to overcome this bias, and thus to obtain a more precise idea of the relative changes of those two 'macro-sectors'; the results of such an empirical study led to the conclusion that, in fact, the phenomenon had been strongly over-estimated by traditional, industry-level, analysis, and that tertiarisation was not a strong tendency of the Italian economy up to that time.

In order to carry out this empirical application, they made use of a linear operator that, being *independent of prices*, applied to a vector of whatever magnitude classified by industries — both in real and nominal terms — could transform it in a vector classified by vertically integrated sectors. Such a linear operator was originally developed by Gossling, first in a paper (Gossling & Dovring 1966) and then in a book (Gossling 1972).

The debate went on for some years. It started with Siniscalco's (1982) article, analysing the productive structure of the Italian economy by industries and sectors, and presenting the Gossling operator; it continued with Rampa's (1985) paper on the study of the industry and the services sectors in Italy in the period 1965-1983; it was then channeled in a book, edited by Pasinetti himself, titled "Structural Change in the Productive System. Integration between Industry and Service Sector"[14] (*Mutamenti strutturali del sistema produttivo. Integrazione tra industria e settore terziario*). The main chapter of the book was the essay by Momigliano and Siniscalco (Momigliano & Siniscalco 1986), followed by a series of comments. Among the others, there was a comment by Giorgio Rampa (on methodological issues: Rampa 1986) and the authors' reply (Siniscalco & Momigliano 1986).

This kind of vertically integrated analysis has been more recently resumed by Montresor and Vittucci Marzetti, again for the study of tertiarisation, considering various groups of OECD countries, with quite interesting results.[15]

As it can be seen, Pasinetti's (1973) formalisation of Sraffa's subsystems has been applied in a relatively small set of empirical works, concerning an even smaller scope of problems. No doubt that traditional input-output models had a much wider application. But the result of these few empirical exercises constitute a clear example of how vertically integrated analysis can give different answers with respect to those which would be obtained by using traditional, industry-level, input-output models.

The potential fields of applications, however, are much more than these.

First of all, measures of changes in (labour) productivity can be computed in terms of vertically *hyper*-integrated sectors also. As maintained elsewhere (Garbellini & Wirkierman 2010a), the effects of technical progress on the production process cannot be summarised by a single measure. A *set* of measures, to be interpreted together in their reciprocal relation, are necessary for a complete understanding of the phenomenon. In this respect, I think that having both vertically integrated and hyper-integrated measures, with their decompositions in direct, indirect, and for the latter, also hyper-indirect labour, can be useful for providing a more complete picture of technical change through time.

The dynamics of technical progress, and thus of labour productivity, also influence international trade relations, according to what Pasinetti has called the "principle of comparative productivity-change advantage" (Pasinetti 1981, p. 266). Therefore, once defined a satisfactory set of vertically integrated and vertically hyper-integrated measures, the same kind of empirical exercise performed in Gar-

[14] This is my own translation of the title of the book which has not been edited in English.

[15] See Montresor & Vittucci Marzetti (2006), Montresor & Vittucci Marzetti (2007a), Montresor & Vittucci Marzetti (2007b) and Montresor & Vittucci Marzetti (2008).

bellini & Wirkierman (2010a) for the case of Italy can be performed for different countries, or groups of countries — and for longer periods of time — in order to analyse the joint dynamics of labour productivity and the patterns of international trade.

Moreover, Pasinetti's (1981) framework, and its analytical generalisation, can also be useful for the study of other aspects of technical progress, such as the process of *capital accumulation* and physical investment — due to the advantage, mentioned above, of allowing to study the dynamics of new investment, and thus of capital accumulation, independently of that of the *composition* of productive capacity — and the dynamics of capital intensity and degree of mechanisation characterising modern industrial systems.

To conclude, all issues related to *dynamics* can be fruitfully studied by using vertical hyper-integration: "over time, the input-output coefficients change and the inter-industry system breaks down. [...] Then it is only the vertically [hyper-]integrated model that allows us to follow the vicissitudes of the economic system through time" (Pasinetti 1981, p. 115). No doubt that there is much work to do in fully generalising the theoretical framework. I have tried to do a first step in Garbellini (2010b) and Garbellini (2010a); it is my conviction that an adequate way of introducing fixed capital into the picture, a complication that I have avoided in this first stage, is necessary to make the model better equipped to suit reality.[16]

## 3.4 Simplifying assumptions

The last two criticisms to Pasinetti's (1981) framework that I want to consider here concern the adoption of simplifying assumptions regarding the description of the technique in use and the laws of movement of the relevant economic magnitudes, respectively.

Let us start from Pasinetti's (1981) description of the technique in use. In the simplified setting of the book, there are $2 \times m$ produced commodities: $m$ consumption commodities and $m$ capital goods. Each consumption commodity $i$ ($i = 1, 2, \ldots, m$) is produced by means of labour and by a specific capital good $k_i$, which enters only that particular production process; i.e. the industry producing capital good $k_i$ provides inputs to the industry producing consumption commodity $i$ only. Capital goods are produced by means of labour alone.[17] Each vertically

[16] It is also worth saying that, unfortunately, it is quite difficult to find proper data from national accounts, be them from the various national statistical offices, from Eurostat, OECD, etc., especially concerning physical capital, with the necessary disaggregation.

[17] I am considering here the 'intermediate case'. Pasinetti (1981) considers also a more complex case, in which capital goods are produced by means of labour and capital goods too. But such a complication does not change in a significant way the description of the

hyper-integrated sector $i$ is therefore made up by two industries ($i$ and $k_i$) that are constituent components of sector $i$ only.

Clearly, this is a crude simplification; the criticism often made is that there are no basic commodities, and no inter-industry relations. However, though being analytically very convenient, this assumption is also *conceptually* very easy to be generalised: each intermediate commodity $k_i$ can be thought of as a particular *composite* commodity, constituted by *all* the physical commodities actually produced in the economic system in different sectoral proportions. The set of intermediate commodities used up — directly, indirectly and hyper-indirectly — for the production of consumption commodity $i$ are a unit of (vertically hyper-integrated) productive capacity, and can be called capital (composite) commodity $k_i$.

The analytical generalisation follows straightforward: by eliminating these simplifying assumptions, the whole set of inter-industry relations is reintroduced into the picture. Each vertically hyper-integrated sector is made up by *all* the industries of the economic system, according to the inputs they provide for the production of the corresponding consumption commodity (see Pasinetti 1988, Garbellini 2010b).

By means of this generalisation, the input-output data coming from national accounts can be fitted into the model and used for empirical applications. All inter-industry relations are taken into account. Each vertically hyper-integrated sector is a *growing* subsystem "repeatedly go[ing] through the whole intricate pattern of inter-industry connections" (Pasinetti 1981, p. 110).

As to the movements through time of the relevant economic magnitudes assumed by Pasinetti (1981), the main argument behind the criticisms can again be summarised by an excerpt taken from Harris's (1982) review of *Structural Change and Economic growth*:

> Pasinetti makes good use of this idea [of the presence of a learning process] on the consumption side of his model. But he does not exploit the full potential on the production side, insofar as he assumes that technical change is a smoothly recurring process taking place at a *constant* (but non-uniform) rate in all sectors.
>
> (Harris 1982, p. 38)

Pasinetti (1981), in sketching his *General multi-sector dynamic model* (Pasinetti 1981, Chapter V) assumes that time is continuous and that all relevant economic magnitudes, namely population, direct labour requirements, and demand

technique in use; moreover, the intermediate case is the one Pasinetti himself considers at length, and it is my contention that it is the most convenient one. For details on this point, see Garbellini & Wirkierman (2010b).

coefficients, change through time exponentially (an assumption borrowed from Harrod 1948) at steady — though different from sector to sector — rates.[18]

The choice of assuming this kind of dynamic movements has been the object of criticisms. The core of the problem lies in the consideration of continuous time. Once this choice is made, it makes not much sense to take non-steady rates of growth. Continuous time was chosen for a matter of simplicity, since it allows to keep the dynamic analysis mathematically as simple as possible, and thus to focus attention on the aspects that Pasinetti (1981) wanted to stress. Introducing the complication of non-steady rates would have made things much more complicated, and therefore the choice of continuous time would have become pointless: "[a]ny other types of movements — continuous or discontinuous — may be hypothesised, though with some obvious complications", (Pasinetti 2007, p. 285n).

The only consistent way of introducing non-steady rates of growth is that of reformulating the whole framework using discrete, rather than continuous, time, which is precisely what I have done in Garbellini (2010a). In this way, the rates of change of the above-mentioned economic magnitudes is different from time period to time period, and a whole series of further consideration can be made concerning dynamics. This clearly is a choice which becomes compulsory, so to speak, when one wants to perform empirical applications using this framework. National accounts data are discrete, not continuous, and the degree of realism — and therefore the possibility of fitting real data — improves if the analytical formulation is made in the same terms.

Before concluding, it is however worth stressing that the choices of the simplifying assumptions made by Pasinetti at that time had very clear reasons. The 1981 book was intended to be the exposition of a new framework for analysing "the dynamics of the wealth of nations". The task was quite ambitious, especially when the great number of issues touched upon by Pasinetti (1981) is taken into account. It was therefore necessary to avoid all possible further complications, in order to make the basic idea and the main results of the book immediately understandable — even in this way, the accomplishment of this objective has not been an easy one. The following passage, though having been written for a different purpose, develops the argument much better than I could do:

[18]Such rates of growth are not arbitrarily fixed, but simply considered as exogenous with respect to the kind of analysis which is carried out *at the fundamental level*. As such, they represent an equal number of *degrees of freedom*, that one can close by using actual data or trying to explain from a theoretical point of view. In principle, therefore, any theoretical or empirically consistent explanation of the behaviour of such rates of growth can be introduced at the 'institutional level'. The meaning of the term 'pre-institutional', and therefore the scope of foundational analysis as opposed to that of institutional one, has been analysed and discussed in Garbellini & Wirkierman (2010b, section 2).

> [T]he economists of early centuries set themselves the rather ambitious task of studying economic reality in all its complexity, using, however, somewhat crude methods of analysis.
>
> Today economists are more conscious of the complexity of real economic relationships and adopt the procedure of initially assuming a simplified economic system. This simplified economic system is, however, studied in a rigorous way, with analytical methods which, in principle at least, should leave no room for any ambiguity. It is only after having studied a simplified economic system that the attempt is then made to introduce, one at a time, more complex hypotheses. This procedure is of course followed in the present analysis.
>
> (Pasinetti 1977, p. 35)

The task of studying a simplified economic system has been accomplished in an excellent way by Pasinetti (1981); the introduction of a first set of more complex hypotheses has been achieved by Pasinetti (1988). I hope to have been able to do a further step forward with Garbellini (2010b) and Garbellini (2010a). The remainder of the path is still awaiting for future research.

# Conclusions

As stated in the Introduction, the present dissertation is intended to be a preparatory, *theoretical* work paving the way for institutional analysis and empirical applications.

Pasinetti's (1981) original theoretical framework, developing the *foundational stage* of the analysis, still needs at least one further step to be completely generalised: the treatment of *fixed capital*. However, it is my contention — even though, for the time being, still at the stage of an intuition — that the device of vertical hyper-integration allows this generalisation to be performed in a quite straightforward way, without necessarily going through fixed capital as a joint product.

I would also like to add that I am aware of the fact that some issues faced in the dissertation, and especially in the second and third chapters, would deserve a deeper treatment and a more careful analysis. However, I have tried here at least to mention the results I considered as being relevant, though sometimes without the possibility of fully developing them.

In order to implement empirical applications, then, it will be necessary to scrutinise the relation between the nature of the data provided by national accounts and the building categories of the theoretical framework itself. This further preliminary work is, in my opinion, necessary for having a clearcut idea of the consequences of data limitations, and therefore for trying to overcome them. It is a matter of fact that as time goes by, national accounting practice becomes more and more influenced by, and directed to meet, the necessities of 'mainstream' economics.

Moreover, the lack of inter-industry matrices in physical terms compels to look for proper deflation methods aiming at avoiding as much as possible the loss of information, and the distortion of data. Clearly, once constant price matrices are computed, the kind of empirical analyses that can be performed have as their object *changes through time*; this makes the necessity of a framework allowing to deal with dynamics, such as the device of vertical hyper-integration, even more binding.

Coming to the possible lines of further research, I would like to mention at least three of them, which I consider particularly interesting and which I myself would like to pursue in the future, though with the awareness that institutional analysis is opened up to cover a much wider scope of investigation.

First, I am thinking of the reappraisal of a research field which has been very fruitful especially in the Seventies and in the Eighties: the analysis of changes in labour productivity. In the third chapter of the present dissertation I have provided a series of different decompositions of each sector's vertically hyper-integrated labour. It can be decomposed as the sum of direct, indirect, and hyper-indirect labour; but also as the sum of vertically integrated labour for the production of the final consumption commodity and of vertically integrated labour for the production of the corresponding additional productive capacity. Moreover, through series expansion of the inverse matrices $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ $(i = 1, 2, \ldots, m)$ it can be seen as the sum of vertically hyper-integrated labour for producing final consumption commodity $i$, plus vertically hyper-integrated labour for producing the corresponding productive capacity, plus vertically hyper-integrated labour for the production of productive capacity for vertically hyper-integrated productive capacity itself, and so on.

All these decompositions might be the basis for computing different measures accounting for changes in labour productivity: as stated in Garbellini & Wirkierman (2010a, section 5), "it is difficult to accept the usefulness of the search for a unique synthetic index of labour productivity changes that can also describe the structural processes of technical change lying behind them. On the contrary, we think that is very useful to dispose of a set of related measures allowing us to uncover such structural processes".

Closely connected to this first line of applied research, there comes a second one, to be faced first from a theoretical, and then from an applied, point of view. I am thinking of the effects of *technical progress* on the process of growth and capital accumulation, through the analysis of the dynamics of labour productivity, capital intensity, and degree of mechanisation.

Finally, and again in close relationship to the first and second items of this short list, there is a further issue which would be interesting to face first theoretically, through the development of an 'institutional' framework of analysis, and then empirically. I am thinking of the effects that differences in the dynamics of labour productivity, capital intensity, and degree of mechanisation in different countries have on international trade relations. Pasinetti (1981, chapter XI) himself gives interesting insights on how to deal with this issue, deriving what he called "principle of comparative productivity-change advantage" (Pasinetti 1981, p. 266).

To conclude, I would like to quote a passage from the very beginning of Pasinetti's (1981) book. In this passage, after having mentioned a series of, at that time recent, promising contributions by some 'heterodox' economists, Pasinetti stated the most fundamental aim of his theorising:

> At this point of our discussion, it is not difficult to see that all the contributions to economic theory just mentioned stem from what has been called above the production or industry approach to economic reality [...]. But their authors themselves did not perceive this very clearly. Each of those theories have been presented under the compulsion of certain facts, which current theory was unable to explain. As a consequence, they have been presented independently of one another, without an explicit relation to any unifying principle. This has made things easier for the Marginalists. It seemed natural to look for a unifying theoretical framework and marginal economics had one to offer. Although the authors of the new theories have, most of the time, [new page] strongly protested that their theories had nothing to do with Marginalism, the Marginalists have been at an advantage. They have had the advantage of synthesis. For they have always clearly presented their arguments around a unifying problem (optimum allocation of scarce resources) and a unifying principle (the rational process of maximisation under constraints).
>
> Yet it seems to me that it is possible to build a unifying theory behind all the new contributions to economics mentioned above. The foregoing discussion has been constantly pointing towards it. It is a theory the basic elements of which can be traced back to various stages in the development of economic thought; they can be found, here and there, in Smith, in Ricardo, in Malthus, in Marx, in Keynes, in Kalecki, in Leontief, in Sraffa, and in the recent models of economic growth and income distribution. However, these basic elements have not yet been brought out and fitted together in a unifying theoretical scheme. Those economists, who understood remarkably well the requirements of production, did not go into the dynamics of it, which is indeed the aspect that gives it full relevance. The others mainly concentrated on the exploration of particular — though important — aspects, or only on the macro-economic aspects of the process of production in a modern society.
>
> (Pasinetti 1981, pp. 18-19)

I hope I could give, with the present dissertation, my modest contribution.

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